Introduction to Proofs - Sets - Intro

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Define basic terms about sets
2. Define a set using set-builder notation.
3. Distinguish between $\emptyset$ and $\{\emptyset\}$.
4. Prove that two sets are equal using the "double subset technique"

Motivation

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Motivation 2

Sets are a fundamental way of encoding math. We can encode lists, numbers and functions all from only sets.

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- $\{x, y, z\}$.
- $\{\text{Mike}, \text{Qun}\}$
- $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$.
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Observations:

1. Order doesn't matter.
2. Repeats don't matter.
3. Sets can contain objects of any type (including other sets!).

Elements

Definition ($x \in A$)

If x is an object, and A is a set, we say $\underline{x \in A}$ if x is an element (or member) of A. We say $y \notin A$ if y is not an element of A.

Examples:

1. $1 \in \{1, 2, 5\}$ and $3 \notin \{1, 2, 5\}$
2. $-1 \in \mathbb{Z}$ and $-1 \notin \mathbb{N}$
3. $\{0, 1\} \in \{1, 7, \{0, 1\}\}$ and $0 \notin \{1, 7, \{0, 1\}\}$.

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Convention

We prefer to use upper case letters (A, B, C, X, Y) for sets, and lower case letters (a, b, c, x, y) for elements.

Subsets

Definition (subset)

Let A, B be sets. We say that $A \subseteq B$ if and only if $(\forall x)[x \in A \implies x \in B]$.

Examples

1. $\{1, 7\} \subseteq \{0, 1, 2, 3, 7\}$.
2. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Non-examples

1. $\{-1, 1\} \not\subseteq \mathbb{N}$.

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Negation of $A \subseteq B$

$\neg(A \subseteq B)$ means $(\exists x)[x \in A \wedge x \notin B]$

Lemmas about subsets

Lemmas

Let A, B, C be sets.

1. $A \subseteq A$.
2. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Exercise: Prove those two statements directly (by definition unwinding).

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How many elements does the set $B = \{0, 1, A\}$ have?

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2. If $A = 2$, then
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1. If $A = 0$, then B has two elements: 0 and 1.
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2. If $A = 2$, then B has three elements: $0, 1$ and 2.
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1. If $A = 0$, then B has two elements: 0 and 1.
2. If $A = 2$, then B has three elements: $0, 1$ and 2.
3. If $A = \{0, 1\}$, then B has three elements: $0, 1$, and $\{0, 1\}$.

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Theorem

$\emptyset \neq \{\emptyset\}$.

Set builder notation

Definition (Set builder notation)

If A is a set, and $P(x)$ is a property of x, then

$$\{x \in A : P(x)\}$$

is the set of all $x \in A$ such that $P(x)$ is true.

Example:

1. $\{x \in \mathbb{Z} : 1 \leq x < \pi\} = \{1, 2\}$.

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Important rewording

$y \in \{x \in A : P(x)\} \Leftrightarrow (y \in A \wedge$ "$P(y)$ is true".)

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To show $A = B$ (where A, B are sets), you need to show:

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Warning: Avoid using the alternate technique if you are lazy, since you need to check that every $\Leftrightarrow$ is not just a $\Rightarrow$.

Exercises

1. Write out all elements of $\{x \in \mathbb{Z} : x^2 - 1 < 3\}$.
2. Use the definition of subset to prove that $\{1, 2\} \subseteq \{0, 1, 2, 3\}$.
3. Express the even integers using set-builder notation.
4. Let A be a set. Show that $\emptyset \subseteq A$.
5. Give an example of sets A, B such that $A \in B$ and $A \subseteq B$ are both true.

Reflection

- What is the difference between $x \in A$ and $A \subseteq B$?
- Think of a real life example of a set with a subset.
- Is it possible for $A \subseteq B$ and $B \subseteq A$? What about $x \in A$ and $A \in x$?
- Is $\emptyset = \{\emptyset\}$? Why or why not?