Introduction to Proofs - Sets - Constructions

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Use interval notation to describe a set.
2. Give the definitions of set intersection, union, difference, complement and product.
3. Identify mathematical statements involving intersection, union, difference, complement and product.

Motivation

Motivation 1

Intervals are important, simple subsets of $\mathbb{R}$.

Motivation 2

We will construct new sets from old sets using common operations.
(These will be related to the logical operations $\wedge, \vee, \implies$, etc.)

Interval definitions

Definitions (Intervals)

Let $a, b \in \mathbb{R}$. The intervals are defined as follows:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, an open interval.
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, a closed interval.
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, a half-open interval.
- $(a, \infty) = \{x \in \mathbb{R} : a < x\}$, an open ray.
- $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$, a closed ray.

Note: Many other combinations are possible (open/closed, ray/interval, half, etc.).

Set Operations

We will go through five basic operations:

1. Intersection, $A \cap B$
2. Union, $A \cup B$
3. Difference, $A \setminus B$
4. Complement, A^c
5. Cartesian Product, $A \times B$

Intersection

Definition (Intersection)

Let A, B be sets. The intersection of A and B is defined as:

$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

Equivalently, $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$.

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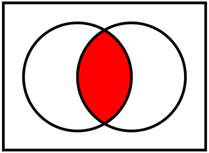
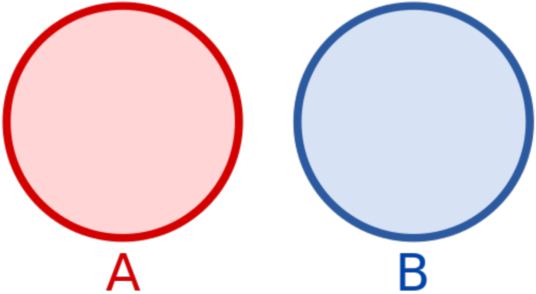


Example: $\mathbb{N} \cap (1, \pi] = \{2, 3\}$.

Special case. When $A \cap B = \emptyset$, we say A and B are disjoint. (They don't share any elements.)
Example: $\mathbb{N}$ and $(-\infty, 0]$ are disjoint sets.



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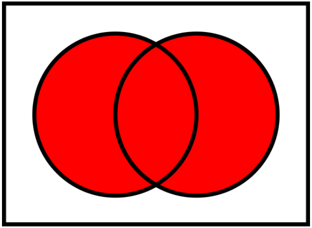

Example: $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$.

Exercise

Check-in:

- Is the logical operator $\wedge$ related to $\cap$ or $\cup$? How?
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$$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$$

$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$$

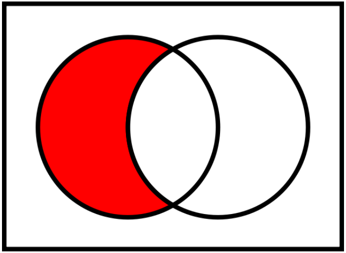
Difference

Definition (Difference)

Let A, B be sets. The set difference of A remove B is defined as:

$$A \setminus B = \{x : x \in A \wedge x \notin B\}.$$

Equivalently, $x \in A \setminus B \Leftrightarrow (x \in A \wedge x \notin B)$.



Example: $[1, 3] \setminus \{2\} = [1.2) \cup (2, 3]$.

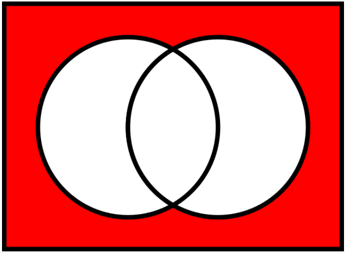
Complement

Definition (Complement)

Let A, U be sets. The complement of A (with respect to the universal set U) is defined as:

$$A^c = \{x \in U : x \notin A\}.$$

Equivalently, $x \in A^c \Leftrightarrow (x \in U \wedge x \notin A)$.



Example: For $U = \mathbb{R}$, $(1, 3]^c = (-\infty, 1] \cup (3, \infty)$.

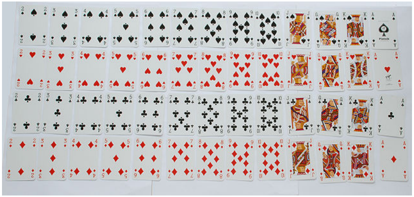
Cartesian Product

Definition (Cartesian Product)

Let A, B be sets. The Cartesian product of A and B is defined as:

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}.$$

Equivalently, $(x, y) \in A \times B \Leftrightarrow (x \in A \wedge y \in B)$.



An illustration of $A \times B$ with and $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, \text{Ace}\}$ and $B = \{\text{Diamonds, Clubs, Hearts, Spades}\}$, where for example, (K, Spades) is the king of spades.
Used with permission of Trainler, https://en.wikipedia.org/wiki/File:Piatnikcards.jpg

Cartesian products, part 2

1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R} \wedge y \in \mathbb{R}\}$, it is the xy plane.

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Exercise

Which of the following sets is $(1, 3)$ an element of?

1. $[0, 2] \times \mathbb{R}$.
2. $([0, 2] \cap \mathbb{N}) \times \{2, 3\}$.
3. $[0, 2] \times [2, 4]$.

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Answer: All of them!

Reflection

- Is the intersection of two intervals always an interval? What about the union? (Prove it.)
- How can $A \setminus B$ be defined in terms of intersections, unions and complements?
- Is $A \cup B = B \cup A$? What logical identity/fact does this amount to?
- If A and B are finite, then find a formula for the number of elements of $A \times B$.