Introduction to Proofs - Sets - Identities and Counterexamples

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Identify plausible set identities.
2. Identify plausible locations for counterexamples to a false set identity.
3. Prove a set identity using the double subset technique.

Motivation

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How can we identify when a set identity is true or false?

Example: Which of the following statements are true for all sets A, B, C?

1. $A \cap B \subseteq A$
2. $A \setminus B = B \setminus A$
3. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Counterexample

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Is $A \cup (B \setminus C) = (A \cup B) \setminus C$?

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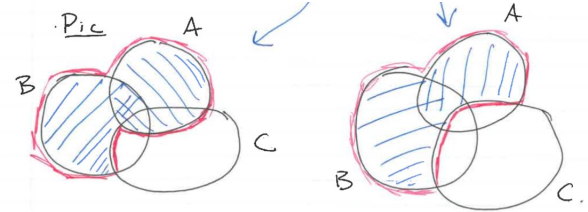

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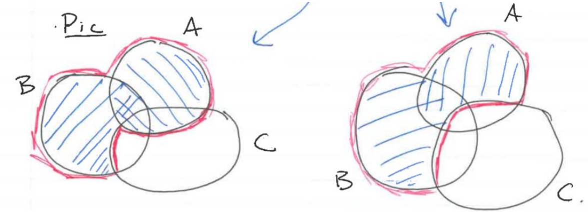

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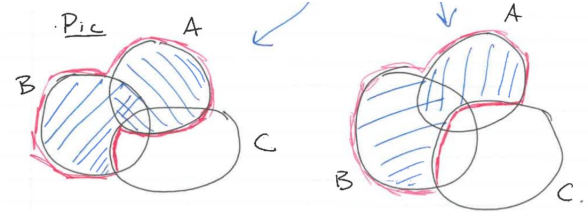

Try $A = B = C = \{7\}$, since $A \cap C \neq \emptyset$.
$A \cup (B \setminus C) = A \cup \emptyset = \{7\}$ but $(A \cup B) \setminus C = \{7\} \setminus \{7\} = \emptyset$.

Double Subset

Theorem

For all sets A, B, C we have $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

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We use the double subset technique. First we prove "$\subseteq$".

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We use the double subset technique. First we prove "$\subseteq$".
Let $z \in A \times (B \cup C)$.

So $z \in (A \times B) \cup (A \times C)$.

□

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For all sets A, B, C we have $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof.

We use the double subset technique. First we prove "$\subseteq$".
Let $z \in A \times (B \cup C)$.

So $z \in A \times B \vee z \in A \times C$.
So $z \in (A \times B) \cup (A \times C)$.

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Let $z \in A \times (B \cup C)$.

There is an $x \in A$ and $y \in (B \cup C)$ such that $z = (x, y)$.

So $z \in A \times B \vee z \in A \times C$.

So $z \in (A \times B) \cup (A \times C)$.

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Let $z \in A \times (B \cup C)$.
There is an $x \in A$ and $y \in (B \cup C)$ such that $z = (x, y)$.
So $x \in A \wedge (y \in B \vee y \in C)$.

So $z \in A \times B \vee z \in A \times C$.
So $z \in (A \times B) \cup (A \times C)$.

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So $x \in A \wedge (y \in B \vee y \in C)$.
So $(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$.
So $z \in A \times B \vee z \in A \times C$.
So $z \in (A \times B) \cup (A \times C)$.

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There is an $x \in A$ and $y \in (B \cup C)$ such that $z = (x, y)$.

So $x \in A \wedge (y \in B \vee y \in C)$.

So $(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$. As $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$.

So $z \in A \times B \vee z \in A \times C$.

So $z \in (A \times B) \cup (A \times C)$.

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There is an $x \in A$ and $y \in (B \cup C)$ such that $z = (x, y)$.
So $x \in A \wedge (y \in B \vee y \in C)$.
So $(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$. As $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$.
So $z \in A \times B \vee z \in A \times C$.
So $z \in (A \times B) \cup (A \times C)$.
The $\supseteq$ direction is an exercise for you.

□

Set identity using iff

DeMorgan's law for sets

For sets A, B, C we have $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof.

Note that

$$
\begin{aligned}
x \in A \setminus (B \cup C) &\Leftrightarrow \\
&\Leftrightarrow \\
&\Leftrightarrow \\
&\Leftrightarrow \\
&\Leftrightarrow \\
&\Leftrightarrow \\
&\Leftrightarrow
\end{aligned}
$$

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&\Leftrightarrow x \in (A \setminus B) \wedge (x \in A \setminus C) \\
&\Leftrightarrow x \in (A \setminus B) \cap (A \setminus C) && \text{Defn of } X \cap Y
\end{aligned}
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\begin{aligned}
x \in A \setminus (B \cup C) &\Leftrightarrow \\
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&\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\
&\Leftrightarrow x \in (A \setminus B) \wedge (x \in A \setminus C) && \text{Defn of } X \setminus Y \\
&\Leftrightarrow x \in (A \setminus B) \cap (A \setminus C) && \text{Defn of } X \cap Y
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x \in A \setminus (B \cup C) &\Leftrightarrow x \in A \wedge \neg(x \in (B \cup C)) && \text{Defn of } X \setminus Y \\
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&\Leftrightarrow \\
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&\Leftrightarrow x \in (A \setminus B) \wedge (x \in A \setminus C) && \text{Defn of } X \setminus Y \\
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&\Leftrightarrow x \in A \wedge x \notin B \wedge x \notin C && \text{Logic DeMorgan} \\
&\Leftrightarrow x \in A \wedge x \in A \wedge x \notin B \wedge x \notin C && P \equiv P \wedge P \\
&\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) && P \wedge Q \equiv Q \wedge P \\
&\Leftrightarrow x \in (A \setminus B) \wedge (x \in A \setminus C) && \text{Defn of } X \setminus Y \\
&\Leftrightarrow x \in (A \setminus B) \cap (A \setminus C) && \text{Defn of } X \cap Y
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□

Reflection

- How are set identities related to identities in logic?
- What is a way to search for a counterexample systematically, and not randomly?
- What set identity corresponds to the logic identity $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$?