Introduction to Proofs - Functions

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Name and define the parts of a function (domain, codomain, range/image).
2. Represent a function using an arrow diagram.

Motivation

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Functions are basic objects that are closely related to sets. They appear everywhere in math.

Definition of a function

"Definition" (Function)

A function $f : A \to B$ is a "rule" (or "machine") that associates to every element $a \in A$ an element $f(a) \in B$.

- **Input**: Any point from A.
- **Output**: Any point from B.

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Non-example. $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \pm\sqrt{x}$ is <u>not</u> a function, since $f(4) = 2$ and $f(4) = -2$.

Notation

Definition (Domain, Codomain)

If $f : A \to B$ is a function then A is called the domain and B is called the codomain of f.

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Example 1

If $f(x) = \dfrac{1}{x-1}$ we usually mean $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$, or $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{0\}$.

Arrow diagrams

Example 2

Let $g : \{1, 2, 3\} \to \{\text{Mike}, 7, \emptyset\}$ be defined by $g(1) = \text{Mike}$, $g(2) = \text{Mike}$, $g(3) = \emptyset$.
Here $\text{dom}(g) = \{1, 2, 3\}$ and $\text{codom}(g) = \{\text{Mike}, 7, \emptyset\}$

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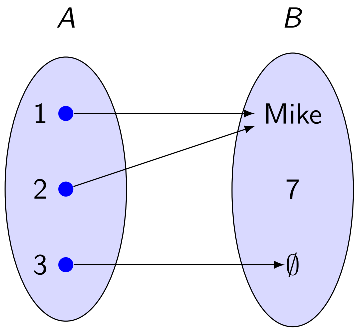

Codomain Example

Example

Consider $f : [-2, 1] \to (-1, 10)$ defined by $f(x) = x^2$. What is:

- $f(1) =$
- $f(-1) =$
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- $f(1) = 1$
- $f(-1) = 1$
- $f(2)$ is not defined, since $2 \notin \text{dom}(f)$.

Range

Definition (Range)

If $f : A \to B$ is a function, then the range (or image) of f is

$$\{f(a) : a \in A\}.$$

Equivalently, $y \in \text{ran}(f) \Leftrightarrow (\exists a \in A)[y = f(a)]$.

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Example 1

Using the function $f : [-2, 1] \to (-1, 10)$ defined by $f(x) = x^2$:

- $\text{dom}(f) = [-2, 1]$.
- $\text{codom}(f) = (-1, 10)$.
- $\text{ran}(f) = [0, 4]$.

Another example

Example 2

Using the function $g : \{1, 2, 3\} \to \{\text{Mike}, 7, \emptyset\}$ defined by $g(1) = \text{Mike}$, $g(2) = \text{Mike}$, $g(3) = \emptyset$.

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- $\text{dom}(g) = \{1, 2, 3\}$.
- $\text{codom}(g) = \{\text{Mike}, 7, \emptyset\}$.
- $\text{ran}(g) = \{g(1), g(2), g(3)\} = \{\text{Mike}, \text{Mike}, \emptyset\} = \{\text{Mike}, \emptyset\}$.

Image of a set

Definition (Image)

If $f : A \to B$ is a function, and $C \subseteq A$, then the image of C under f is the set:

$$f(C) = \{f(c) : c \in \mathbb{C}\}.$$

Example

If $g : \mathbb{N} \to \mathbb{R}$ is defined by $g(x) = (-1)^x$, and E is the collection of even naturals, then

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Observations

Propositions

If $f : A \to B$ is a function, and $C \subseteq A$, then

1. $\text{ran}(f) \subseteq \text{codom}(f)$.
2. $f(C) \subseteq \text{codom}(f)$
3. $f(C) \subseteq \text{ran}(f)$
4. $\text{ran}(f) = f(\text{dom}(f))$

Why use codomain?

Observation

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You want to define a function `Country(x)` that takes as an input x a year in which the Olympics were held, and outputs the name of the country that won the most medals that year.

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- ran($\texttt{Country}$) $= \ldots$

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- ran($\texttt{Country}$) = ... requires research to find out.

Reflection

- What are the key parts of a function?
- If a function is given to you as $f : A \to B$, what two features can you identify already?
- What is the difference between the codomain and range of a function?