Introduction to Proofs - Equivalence Relations

Prof Mike Pawliuk

UTM

June 4, 2020

Slides available at: mikepawliuk.ca

Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Check if a given relation is transitive, symmetric, or reflexive.

Motivation

Motivation

Often things are equal in some ways, but different in others. Equivalence relations capture this idea.

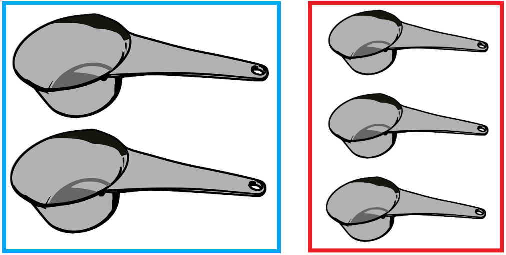
Motivation

Motivation

Often things are equal in some ways, but different in others. Equivalence relations capture this idea.

Example

The numbers $\frac{2}{2}$ and $\frac{3}{3}$ are different representations, but equivalent amounts.



This image is used with permission from Pixabay.
https://pixabay.com/vectors/cup-kitchen-measure-measuring-161133/

Definitions

Definition

A relation R on a set X is

- Reflexive if $(\forall x \in X)[(x, x) \in R]$,

Definitions

Definition

A relation R on a set X is

- Reflexive if $(\forall x \in X)[(x, x) \in R]$,
- Symmetric if $(\forall x, y \in X)[(x, y) \in R \implies (y, x) \in R]$

Definitions

Definition

A relation R on a set X is

- Reflexive if $(\forall x \in X)[(x,x) \in R]$,
- Symmetric if $(\forall x,y \in X)[(x,y) \in R \implies (y,x) \in R]$
- Transitive if $(\forall x,y,z \in X)[(x,y) \in R \wedge (y,z) \in R \implies (x,z) \in R]$

Definitions

Definition

A relation R on a set X is

- Reflexive if $(\forall x \in X)[(x, x) \in R]$,
- Symmetric if $(\forall x, y \in X)[(x, y) \in R \implies (y, x) \in R]$
- Transitive if $(\forall x, y, z \in X)[(x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R]$

A relation R is an equivalence relation if R is reflexive, symmetric, and transitive.

Definitions

Definition

A relation R on a set X is

- Reflexive if $(\forall x \in X)[(x,x) \in R]$,
- Symmetric if $(\forall x, y \in X)[(x,y) \in R \implies (y,x) \in R]$
- Transitive if $(\forall x, y, z \in X)[(x,y) \in R \wedge (y,z) \in R \implies (x,z) \in R]$

A relation R is an equivalence relation if R is reflexive, symmetric, and transitive.

Intuition: Equivalence relations capture some notion of "sameness".

Definitions

Definition

A relation R on a set X is

- Reflexive if $(\forall x \in X)[(x, x) \in R]$,
- Symmetric if $(\forall x, y \in X)[(x, y) \in R \implies (y, x) \in R]$
- Transitive if $(\forall x, y, z \in X)[(x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R]$

A relation R is an equivalence relation if R is reflexive, symmetric, and transitive.

Intuition: Equivalence relations capture some notion of "sameness".

Negation

These properties are all universal properties. To show that a relation fails one of the properties you need to produce a counterexample.

Non-example 1

Non-example 1

Let $X = \mathbb{R}$, $R_1 = \{(x, y) \in \mathbb{R}^2 : x < y\}$.

Non-example 1

Non-example 1

Let $X = \mathbb{R}$, $R_1 = \{(x, y) \in \mathbb{R}^2 : x < y\}$.

Not reflexive. $1 \in \mathbb{R}$ and $(1, 1) \notin R_1$ as $1 \not< 1$.

Non-example 1

Let $X = \mathbb{R}$, $R_1 = \{(x, y) \in \mathbb{R}^2 : x < y\}$.

Not reflexive. $1 \in \mathbb{R}$ and $(1, 1) \notin R_1$ as $1 \not< 1$.

Not symmetric. $(1, 2) \in R_1$ and $(2, 1) \notin R_1$ as $2 \not< 1$.

Non-example 1

Non-example 1

Let $X = \mathbb{R}$, $R_1 = \{(x, y) \in \mathbb{R}^2 : x < y\}$.

Not reflexive. $1 \in \mathbb{R}$ and $(1, 1) \notin R_1$ as $1 \not< 1$.

Not symmetric. $(1, 2) \in R_1$ and $(2, 1) \notin R_1$ as $2 \not< 1$.

Transitive. Let $(x, y) \in R_1$ and $(y, z) \in R_1$. So $x < y$ and $y < z$. So $x < z$ (by a property of $<$). So $(x, z) \in R_1$.

Non-example 2

Non-example 2

Let $X = \mathbb{R}$, $R_2 = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$.

Non-example 2

Non-example 2

Let $X = \mathbb{R}$, $R_2 = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$.

Reflexive. Let $x \in \mathbb{R}$. Note that $x \leq x$, so $(x, x) \in R_2$.

Non-example 2

Non-example 2

Let $X = \mathbb{R}$, $R_2 = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$.

Reflexive. Let $x \in \mathbb{R}$. Note that $x \leq x$, so $(x, x) \in R_2$.

Not symmetric. Same reason as Example 1.
Transitive. Basically same reason as Example 1.

Non-example 3

Non-example 3

Let $X = \mathbb{Q}$, $R_3 = \{(x, y) \in \mathbb{Q}^2 : |x - y| < 3\}$.

Non-example 3

Let $X = \mathbb{Q}$, $R_3 = \{(x, y) \in \mathbb{Q}^2 : |x - y| < 3\}$.

Reflexive. Let $x \in \mathbb{Q}$. Note that $|x - x| = |0| = 0 < 3$, so $(x, x) \in R_3$.

Non-example 3

Let $X = \mathbb{Q}$, $R_3 = \{(x, y) \in \mathbb{Q}^2 : |x - y| < 3\}$.

Reflexive. Let $x \in \mathbb{Q}$. Note that $|x - x| = |0| = 0 < 3$, so $(x, x) \in R_3$.

Symmetric. Let $(x, y) \in R_3$. So $|x - y| < 3$, and since $|y - x| = |-(x - y)| = |x - y|$ we have $|y - x| < 3$. So $(y, x) \in R_3$

Non-example 3

Let $X = \mathbb{Q}$, $R_3 = \{(x, y) \in \mathbb{Q}^2 : |x - y| < 3\}$.

Reflexive. Let $x \in \mathbb{Q}$. Note that $|x - x| = |0| = 0 < 3$, so $(x, x) \in R_3$.

Symmetric. Let $(x, y) \in R_3$. So $|x - y| < 3$, and since $|y - x| = |-(x - y)| = |x - y|$ we have $|y - x| < 3$. So $(y, x) \in R_3$

Not Transitive. Note that $|1 - 3| = 2 < 3$ and $|3 - 5| = 2 < 3$ (so $(1, 3) \in R_3$ and $(3, 5) \in R_3$) but $|1 - 5| = 4 \not< 3$, so $(1, 5) \notin R_3$.

Examples

Example 1

Let $X = \mathbb{Z}$, $R = \{(x, y) \in \mathbb{Z}^2 : |x| = |y|\}$.

Examples

Example 1

Let $X = \mathbb{Z}$, $R = \{(x, y) \in \mathbb{Z}^2 : |x| = |y|\}$.

Reflexive. Let $x \in \mathbb{Z}$. Note that $|x| = |x|$, so $(x, x) \in R$.

Examples

Example 1

Let $X = \mathbb{Z}$, $R = \{(x, y) \in \mathbb{Z}^2 : |x| = |y|\}$.

Reflexive. Let $x \in \mathbb{Z}$. Note that $|x| = |x|$, so $(x, x) \in R$.

Symmetric. Let $(x, y) \in R$. So $|x| = |y|$, and so $|y| = |x|$. So $(y, x) \in R$.

Examples

Example 1

Let $X = \mathbb{Z}$, $R = \{(x, y) \in \mathbb{Z}^2 : |x| = |y|\}$.

Reflexive. Let $x \in \mathbb{Z}$. Note that $|x| = |x|$, so $(x, x) \in R$.

Symmetric. Let $(x, y) \in R$. So $|x| = |y|$, and so $|y| = |x|$. So $(y, x) \in R$.

Transitive. Let $(x, y) \in R$ and $(y, z) \in R$. So $|x| = |y|$ and $|y| = |z|$. Therefore $|x| = |z|$. So $(x, z) \in R$.

Examples

Example 1

Let $X = \mathbb{Z}$, $R = \{(x, y) \in \mathbb{Z}^2 : |x| = |y|\}$.

Reflexive. Let $x \in \mathbb{Z}$. Note that $|x| = |x|$, so $(x, x) \in R$.

Symmetric. Let $(x, y) \in R$. So $|x| = |y|$, and so $|y| = |x|$. So $(y, x) \in R$.

Transitive. Let $(x, y) \in R$ and $(y, z) \in R$. So $|x| = |y|$ and $|y| = |z|$. Therefore $|x| = |z|$. So $(x, z) \in R$.

Lemma

If $f : X \to Y$ is a function, then $R = \{(x_1, x_2) \in X^2 : f(x_1) = f(x_2)\}$ is a relation on

Example 1

Let $X = \mathbb{Z}$, $R = \{(x, y) \in \mathbb{Z}^2 : |x| = |y|\}$.

Reflexive. Let $x \in \mathbb{Z}$. Note that $|x| = |x|$, so $(x, x) \in R$.

Symmetric. Let $(x, y) \in R$. So $|x| = |y|$, and so $|y| = |x|$. So $(y, x) \in R$.

Transitive. Let $(x, y) \in R$ and $(y, z) \in R$. So $|x| = |y|$ and $|y| = |z|$. Therefore $|x| = |z|$. So $(x, z) \in R$.

Lemma

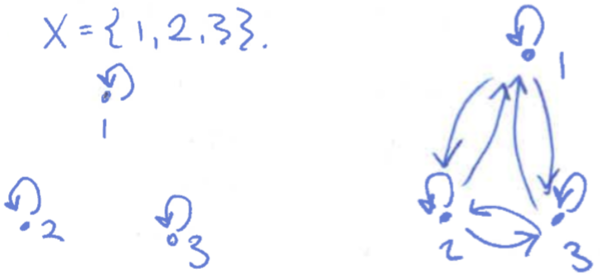
If $f : X \to Y$ is a function, then $R = \{(x_1, x_2) \in X^2 : f(x_1) = f(x_2)\}$ is a relation on X.

Examples

Trivial Examples

Let X be a set. We can always define the following two equivalence relations:

- $E_{\text{singles}} = \{(x, x) : x \in X\}$. (Very few points are related.)
- $E_{\text{full}} = X \times X$. (All points are related.)



For $X = \{1, 2, 3\}$ we have E_{singles} (left) and E_{full} (right).

Examples

Rational representations

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p, q), (x, y)) : py = qx\}$.
Example. $((1, 2), (3, 6)) \in E$ as $1(6) = 2(3)$.

Examples

Rational representations

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p,q),(x,y)) : py = qx\}$.
Example. $((1,2),(3,6)) \in E$ as $1(6) = 2(3)$.
Intuition. $((p,q),(x,y)) \in E$ iff $\dfrac{p}{q} = \dfrac{x}{y}$.

Examples

Rational representations

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p,q),(x,y)) : py = qx\}$.
Example. $((1,2),(3,6)) \in E$ as $1(6) = 2(3)$.
Intuition. $((p,q),(x,y)) \in E$ iff $\dfrac{p}{q} = \dfrac{x}{y}$.

Proposition

This is an equivalence relation.

Proof of Proposition

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p, q), (x, y)) : py = qx\}$.

Proof.

Reflexive and Symmetric are left as exercises for you.
Let $((p, q), (x, y)) \in E$ and $((x, y), (a, b)) \in E$.

Proof of Proposition

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p,q),(x,y)) : py = qx\}$.

Proof.

Reflexive and Symmetric are left as exercises for you.
Let $((p,q),(x,y)) \in E$ and $((x,y),(a,b)) \in E$. So $py = qx$ and $xb = ya$.

Proof of Proposition

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p,q),(x,y)) : py = qx\}$.

Proof.

Reflexive and Symmetric are left as exercises for you.
Let $((p,q),(x,y)) \in E$ and $((x,y),(a,b)) \in E$. So $py = qx$ and $xb = ya$.
Want $((p,q),(a,b)) \in E$. (i.e. $pb = qa$).

Proof of Proposition

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p,q),(x,y)) : py = qx\}$.

Proof.

Reflexive and Symmetric are left as exercises for you.

Let $((p,q),(x,y)) \in E$ and $((x,y),(a,b)) \in E$. So $py = qx$ and $xb = ya$.

Want $((p,q),(a,b)) \in E$. (i.e. $pb = qa$).

Note

$$\begin{aligned} py = qx &\implies pby = bxq \qquad \text{Multiply by } b \\ &\implies \\ &\implies \end{aligned}$$

So $((p,q),(a,b)) \in E$. □

Proof of Proposition

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p,q),(x,y)) : py = qx\}$.

Proof.

Reflexive and Symmetric are left as exercises for you.

Let $((p,q),(x,y)) \in E$ and $((x,y),(a,b)) \in E$. So $py = qx$ and $xb = ya$.

Want $((p,q),(a,b)) \in E$. (i.e. $pb = qa$).

Note

$$\begin{aligned} py = qx &\implies pby = bxq && \text{Multiply by } b \\ &\implies pby = yaq && \text{As } xb = ya \\ &\implies \end{aligned}$$

So $((p,q),(a,b)) \in E$. □

Proof of Proposition

Let $X = \mathbb{Z} \times \mathbb{N}$ and let $E = \{((p,q),(x,y)) : py = qx\}$.

Proof.

Reflexive and Symmetric are left as exercises for you.

Let $((p,q),(x,y)) \in E$ and $((x,y),(a,b)) \in E$. So $py = qx$ and $xb = ya$.

Want $((p,q),(a,b)) \in E$. (i.e. $pb = qa$).

Note

$$\begin{aligned} py = qx &\implies pby = bxq && \text{Multiply by } b \\ &\implies pby = yaq && \text{As } xb = ya \\ &\implies pb = aq && \text{Cancel } y \text{ as } y \in \mathbb{N} \end{aligned}$$

So $((p,q),(a,b)) \in E$. □

Reflection

- To show that a relation is not an equivalence relation, what do you have to show?
- Make precise the idea that "E_{singles} is the smallest equivalence relation on a set" and "E_{full} is the largest equivalence relation on a set".
- Where have you seen equivalence relations before (in this course, or in other math courses)?