Introduction to Proofs - Equivalence Classes

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Identify the equivalence class that an element is in.
2. Partition a space using equivalence classes.

Motivation

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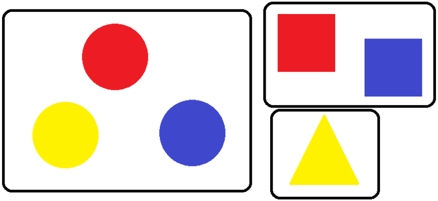
Example

Let X be the set with these 6 coloured shapes, and let E be the equivalence relation "x has the same shape as y".
It has 3 equivalence classes; one for each shape.



Definition

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Let E be an equivalence relation on a set X, and let $a \in X$. The equivalence class of a (with respect to E) is

$$[a] = \{x \in X : (a, x) \in E\}.$$

We also call a a representative for its equivalence class.

Note that $[a] \subseteq X$.

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Example 1

Let $X = \mathbb{Z}$ and let $E = \{(x, y) \in \mathbb{Z}^2 : |x| = |y|\}$.

1. $[-1] = \{-1, 1\} = [1]$
2. $[2] = \{2, -2\} = [-2]$
3. $[0] = \{0\}$

Example 2

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Let $X = \mathbb{Z} \times \mathbb{N}$, and let $E = \{((p,q),(x,y)) : py = xq\}$.

- $[(1,2)] = \{(1,2),(2,4),(3,6),\ldots\}$
- $[(0,1)] = \{(0,n) : n \in \mathbb{N}\}$

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Let $X = \mathbb{Z}$ and let $x \sim y$ if and only if $y - x$ is a multiple of 4.

Exercise. List out all elements of the following equivalence classes:

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Reflect: What do you notice about these equivalence classes?

- Every integer is in at least one equivalence class. (Reflexivity.)
- If an integer is in two equivalence classes, then those classes are the same.

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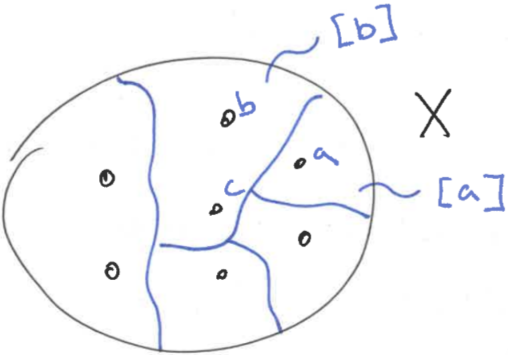
Idea. This theorem says that E "partitions X".



Partition theorem proof

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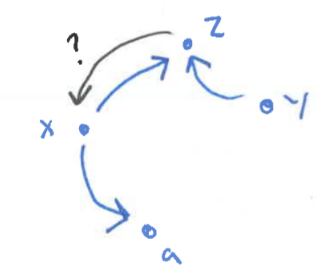

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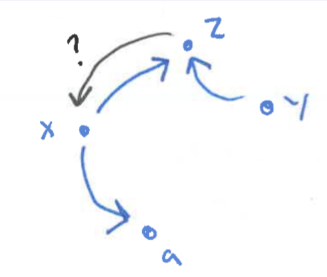

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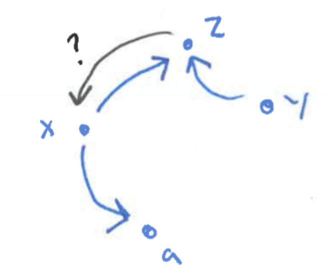

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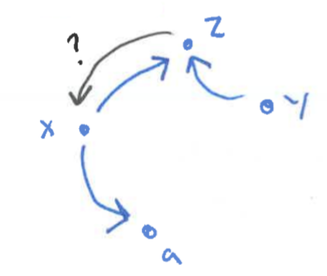

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The other subset direction is similar.

$\square$

Reflection

- What is the type of an equivalence class? (Is it a point, a set, a pair of points, a collection of pairs, or something else?)
- How does the relation $x \sim y$ iff "$y - x$ is a multiple of 4" partition the set $\mathbb{Z}$?
- What is the standard (sometimes called canonical) representative we take for the equivalence class $[(12, 15)]$ for the relation $(p, q) \sim (x, y)$ iff $py = xq$ on the set $\mathbb{Z} \times \mathbb{N}$.