Introduction to Proofs - Inequalities - Axioms

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. State the order axioms.
2. Prove a basic fact using the order axioms.
3. Avoid common pitfalls with inequalities.

Motivation

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We would like to prove a variety of inequalities. How can we do that in a rigourous way?

Notation

Definitions

We say that a real number x is

1. positive, if $0 < x$.
2. negative, if $x < 0$.
3. non-negative, if x is not negative, i.e. $0 \leq x$.

Order Axioms

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Let $a, b, c, d \in \mathbb{R}$.

1. If $a < b$ and $0 < c$, then $ac < bc$.
2. If $a < b$, then $-b < -a$.
3. $a^2 \geq 0$
4. If $a < b$ and $c < d$, then $a + c < b + d$.
5. If $a \geq 0$ then there is a unique non-negative number $\sqrt{a}$ so that $(\sqrt{a})^2 = a$.
6. If $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.
7. If $a < b$ and $b < c$, then $a < c$.

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5. If $a \geq 0$ then there is a unique non-negative number $\sqrt{a}$ so that $(\sqrt{a})^2 = a$.
6. If $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.
7. If $a < b$ and $b < c$, then $a < c$.

Exercise. Show that Axioms 4 is stronger than Axiom 2 (and so we only need to include Axiom 4 on this list). That is, show that you can deduce Axiom 4 from Axiom 2.

Consequences

Facts

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2. If $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

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By Axiom 1 (using $a > 0$) we have $a^2 < ba$.

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Proof of (1).

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Since $ab = ba$, by Axiom 7 we have $a^2 < b^2$. □

Consequences

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Proof of (2).

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Proof of (2).

Assume $0 < a < b$.
By Axiom 4, $0 < b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})$. [*]

Consequences

If $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

Proof of (2).

Assume $0 < a < b$.
By Axiom 4, $0 < b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})$. [*]
Note $0 < \dfrac{1}{\sqrt{b} + \sqrt{a}}$.

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Using Axiom 1 on (*) with $c = \dfrac{1}{\sqrt{b} + \sqrt{a}}$ gives

$$0 \cdot \frac{1}{\sqrt{b} + \sqrt{a}} < (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})\frac{1}{\sqrt{b} + \sqrt{a}}$$

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So $0 < \sqrt{b} - \sqrt{a}$, and by Axiom 4, $\sqrt{a} < \sqrt{b}$. □

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So $0 < \sqrt{b} - \sqrt{a}$, and by Axiom 4, $\sqrt{a} < \sqrt{b}$. □

Exercise. Use the axioms to justify the "note" we made without proof.

Exercise

1. If possible, use the main idea from the proof of fact 1 to prove fact 2.
2. If possible, use the main idea from the proof of fact 2 to prove fact 3.

Common mistakes

Exercise. Each of these are common mistakes. Explain the error that is being made, and how to correct it.

1. If $a < b$, then $-2a < -2b$.
2. Since $0 < 2 < 100$, we have $\frac{1}{2} < \frac{1}{100}$.
3. If x is a real number, then since $2 < 3$ we have $2x < 3x$.
4. If $a < b$, then $a^2 < b^2$.

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3. If x is a real number, then since $2 < 3$ we have $2x < 3x$.
   Error: Since x can be any real number, it could also be negative, and that will change the inequality. (Axiom 2). Break up into 3 cases: $x > 0, x = 0, x < 0$.
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4. If $a < b$, then $a^2 < b^2$.
   Error: Fact 1 requires both a, b to be positive. Can you see why?

Reflection

- What is the difference between an order axiom and a fact?
- Why did we include Axiom 2 on the list of order axioms if it follows from Axiom 4?
- What are some common misunderstandings about inequalities?
- Can a theorem have multiple different proofs?