# Introduction to Proofs - Inequalities - AMGM

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# Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Evaluate the AM and GM of two numbers.
2. Relate/distinguish problem solving and rough work to a proof.

# Motivation

## Motivation 1

We will see two fundamental, non-trivial inequalities in math: the Arithmetic-Mean-Geometric-Mean (AMGM) inequality, and the triangle inequality.
Both inequalities are very geometric, but can be proved entirely from the order axioms.

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Both inequalities are very geometric, but can be proved entirely from the order axioms.

## Motivation 2

The proofs we will see of these inequalities show off a technique of "do rough work, then write out a formal proof".

# Definitions

## Definitions (AM, GM)

Let $a, b \in \mathbb{R}$.

1. The arithmetic mean of $a$ and $b$ is $\dfrac{a+b}{2}$.
2. The geometric mean of $a$ and $b$ is $\sqrt{ab}$ (when it is defined).

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3. **Example 3**. If $a = b$, then the AM is $\frac{a+a}{2} = a$ and the GM is $\sqrt{a \cdot a} = |a|$.

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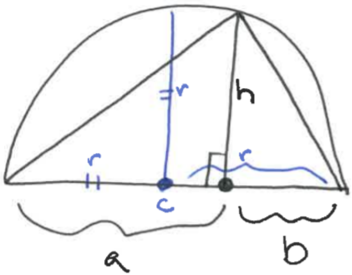
3. **Example 3**. If $a = b$, then the AM is $\dfrac{a+a}{2} = a$ and the GM is $\sqrt{a \cdot a} = |a|$.
4. **Example 4**. If $b = 0$, then the AM is $\dfrac{a}{2} = a$ and the GM is $\sqrt{a \cdot 0} = 0$.

# Geometric motivation




1. Make a circle whose diameter is $a + b$.

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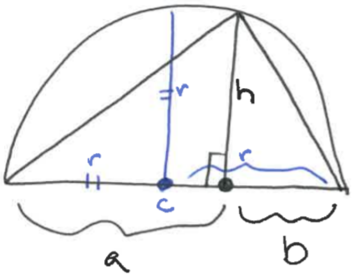
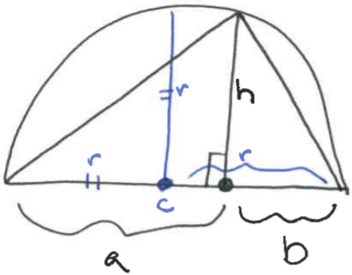



1. Make a circle whose diameter is $a + b$.
2. The radius will be the AM.

# Geometric motivation

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3. The vertical line segment $h$, will be the GM. (This requires proof.)

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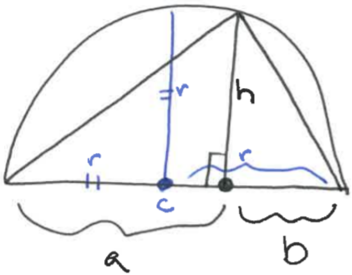


1. Make a circle whose diameter is $a + b$.
2. The radius will be the AM.
3. The vertical line segment $h$, will be the GM. (This requires proof.)
4. The GM is clearly less than or equal to the AM, and equality happens only when $a = b$.

# Textbook proof of AMGM

## Theorem

Let $a, b \in \mathbb{R}$. Then $\sqrt{ab} \leq \frac{a+b}{2}$ (when the GM is defined).

Alternatively, $ab \leq \left(\frac{a+b}{2}\right)^2$.

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$$ab \leq \left(\frac{a+b}{2}\right)^2 \implies ab \leq \frac{a^2+2ab+b^2}{4}$$

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$$ab \leq \left(\frac{a+b}{2}\right)^2 \implies ab \leq \frac{a^2+2ab+b^2}{4}$$

$$\implies 4ab \leq a^2+2ab+b^2$$

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$$\begin{aligned} ab \leq \left(\frac{a+b}{2}\right)^2 &\implies ab \leq \frac{a^2+2ab+b^2}{4} \\ &\implies 4ab \leq a^2+2ab+b^2 \\ &\implies 0 \leq a^2-2ab+b^2 \\ &\implies 0 \leq (a-b)^2 \checkmark \end{aligned}$$

# Proof of AMGM

## Theorem

Let $a, b \in \mathbb{R}$. Then $ab \leq \left(\frac{a+b}{2}\right)^2$.

## Proof.

Let $a, b \in \mathbb{R}$. Note that $0 \leq (a-b)^2$.

$$\begin{aligned} 0 \leq (a-b)^2 &\implies 0 \leq a^2 - 2ab + b^2 \\ &\implies 4ab \leq a^2 + 2ab + b^2 \\ &\implies \implies ab \leq \frac{a^2 + 2ab + b^2}{4} \\ &\implies ab \leq \left(\frac{a+b}{2}\right)^2 \end{aligned}$$

□

# Is this a proof?

"Proof" 1.

$7 \leq 1 \implies 0 \cdot 7 \leq 0 \cdot 1 \implies 0 \leq 0 \checkmark$. So $7 \leq 1$. □

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**"Proof" 2.**

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\begin{aligned}
1 + 2 + 4 + 8 + 16 &< 32 \\
7 + 24 &< 32 \\
31 &< 32 \checkmark
\end{aligned}
$$

□

# Is this a proof?

**“Proof” 1.**

$7 \leq 1 \implies 0 \cdot 7 \leq 0 \cdot 1 \implies 0 \leq 0 \checkmark$. So $7 \leq 1$. $\square$

**“Proof” 2.**

$$\begin{aligned} 1+2+4+8+16 &< 32 \\ 7+24 &< 32 \\ 31 &< 32 \checkmark \end{aligned}$$

$\square$

**Common mistake**

Proofs must begin with something that is true or assumed, and derive what you want. You cannot start with what you want to prove, and then deduce something true.

# Should I include my ugly garbage in my beautiful solutions?

## Rough work / Ugly Garbage

$$
\begin{aligned}
ab \le \left(\frac{a+b}{2}\right)^2 &\implies ab \le \frac{a^2+2ab+b^2}{4} \\
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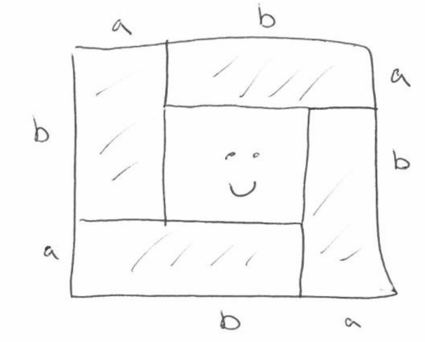
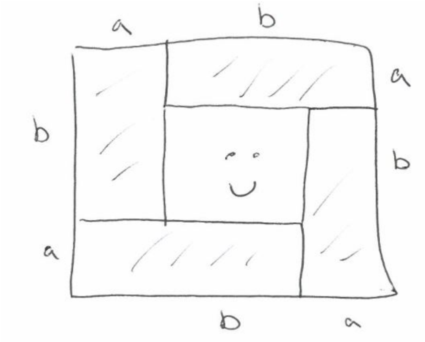
$\square$

# Picture proof of AMGM



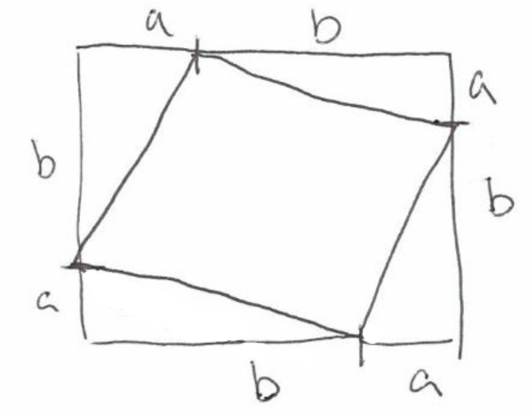

1. The total area is $(a+b)^2$.

# Picture proof of AMGM




1. The total area is $(a+b)^2$.
2. There are four rectangles with area $ab$ each. (so $4ab$ total).
3. Clearly $4ab \leq (a+b)^2$.

# Challenge




**Behold!** This is the proof of a famous theorem. Which one?

# Reflection

- How will you remember whether it's AM $\leq$ GM or GM $\leq$ AM?
- Do geometric proofs count as "real" proofs?
- What is the value of "picture proofs"?
- Which proof of the AMGM do you prefer? (The circle proof, the algebra proof, or the rectangle proof?)