Introduction to Proofs - Triangle Inequality

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Slides available at: mikepawliuk.ca
By the end of this video, participants should be able to:

1. Apply the absolute value function in various contexts (algebraically and geometrically).
2. Relate/distinguish problem solving and rough work to a proof.
3. Break up a sum using the triangle inequality.
“The sum of any two sides of a triangle is always greater or equal to the third side.”

The triangle inequality is one of the two important inequalities we see in this course. We will understand it geometrically and algebraically.
Definition (Absolute value)

If $x$ is a real number, then

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$
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If $x$ is a real number, then

$$|x| = \begin{cases} 
  x & x \geq 0 \\
  -x & x < 0 
\end{cases}$$

**Special case.** $|a - b|$ is the distance between $a$ and $b$. 

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[Diagram showing the distance between points $a$ and $b$ on a number line.]
This picture shows us the inequality: for all real $x$, $x \leq |x|$. 
More visualizations of absolute value

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What is the max value of $|x^2 - 1|$ when $|x| \leq 1$?
More visualizations of absolute value

This picture shows us the inequality: for all real $x$, $x \leq |x|$.

What is the max value of $|x^2 - 1|$ when $|x| \leq 1$? At $x = 0$, $|x^2 - 1| = 1$. 
Basic Facts about absolute value

**Facts**

Let $x, y \in \mathbb{R}$.

1. $x^2 = |x|^2$
2. $\sqrt{x^2} = |x|
3. $x \leq |x|
4. $|xy| = |x||y|$
Basic Facts about absolute value

**Facts**

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3. $x \leq |x|
4. $|xy| = |x||y|

**Special case of (4).** If $y = -1$, get $| - x| = |x|\cdot | - 1| = |x|$. 
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Special case of this. $|b - a| = |- (a - b)| = |a - b|$. 
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Let $x, y \in \mathbb{R}$.

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3. $x \leq |x|$
4. $|xy| = |x||y|$

Special case of (4). If $y = -1$, get $| - x| = |x|| -1| = |x|$.

Special case of this. $|b - a| = | -(a - b)| = |a - b|$.

The distance from $a$ to $b$ is the same as the distance from $b$ to $a$. 
Motivation for Triangle Inequality

\[ |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| \]

**Intuition.** Direct routes are shorter than detours (to z).
Theorem (Triangle Inequality)

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$. 

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Triangle Inequality - Rough work
Theorem (Triangle Inequality)
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Rough work / Ugly garbage

$$|x + y| \leq |x| + |y|$$
**Theorem (Triangle Inequality)**

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

**Rough work / Ugly garbage**

\[
|x + y| \leq |x| + |y| \\
\implies |x + y|^2 \leq (|x| + |y|)^2
\]
Theorem (Triangle Inequality)
For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$. 

Rough work / Ugly garbage

\[ |x + y| \leq |x| + |y| \]
\[ \Rightarrow |x + y|^2 \leq (|x| + |y|)^2 \]
\[ \Rightarrow (x + y)^2 \leq |x|^2 + 2|x||y| + |y|^2 \]
Triangle Inequality - Rough work

**Theorem (Triangle Inequality)**
For all \( a, b \in \mathbb{R} \), \(|a + b| \leq |a| + |b|\).

**Rough work / Ugly garbage**

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|x + y| \leq |x| + |y|
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\implies (x + y)^2 \leq |x|^2 + 2|x||y| + |y|^2
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\[
\implies x^2 + 2xy + y^2 \leq |x|^2 + 2|xy| + |y|^2
\]

This is the basic fact we will start from.
Theorem (Triangle Inequality)
For all \( a, b \in \mathbb{R} \), \(|a + b| \leq |a| + |b|\).

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\Rightarrow 2xy \leq 2|xy|
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For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

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Theorem (Triangle Inequality)

For all \( a, b \in \mathbb{R} \), \(|a + b| \leq |a| + |b|\).

Rough work / Ugly garbage

\[ |x + y| \leq |x| + |y| \]
\[ \implies |x + y|^2 \leq (|x| + |y|)^2 \]
\[ \implies (x + y)^2 \leq |x|^2 + 2|x||y| + |y|^2 \]
\[ \implies x^2 + 2xy + y^2 \leq |x|^2 + 2|xy| + |y|^2 \]
\[ \implies 2xy \leq 2|xy| \]
\[ \implies xy \leq |xy| \leftarrow \text{This is the basic fact we will start from.} \]
Proof.

Let $x, y \in \mathbb{R}$.

Note that $xy \leq |xy|$ by Fact 3.

So $2xy \leq 2|xy| = 2|x||y|$.

Since, by Fact 1, $x^2 = |x|^2$ and $y^2 = |y|^2$ we get

$$x^2 + 2xy + y^2 \leq |x|^2 + 2|xy| + |y|^2.$$  

Factoring gives $|x + y|^2 \leq (|x| + |y|)^2$. By Fact 2 of inequalities, $|x + y| \leq ||x| + |y||$.

Note that $||x| + |y|| = |x| + |y|$, since $|x| \geq 0$, $|y| \geq 0$ and $|x| + |y| \geq 0$.

So $|x + y| \leq |x| + |y|$. 

\[\square\]
Example

Find an $M$ so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$. 

Proof. Solution Use the Triangle inequality.

\[
|x^5 - 2x - 5| = |x^5| + |-(2x + 5)| \\
\leq |x| + |2x + 5| \\
\leq |x| + 2|x| + 5 \\
\leq 2 + 2 \cdot 2 + 5 = 41 \\
\text{since } |x| \leq 2
\]

So take $M = 41$. 

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Example

Find an $M$ so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$.

Proof.

Solution Use the Triangle inequality.

$$|x^5 - 2x - 5| = |x^5 + (-2x - 5)|$$
Example

Find an $M$ so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$.

Proof.

Solution Use the Triangle inequality.

$$|x^5 - 2x - 5| = |x^5 + (-2x - 5)|$$

$$\leq |x^5| + |-2x - 5|$$

By Triangle ineq

So take $M = 41$. 
**Example**

Find an $M$ so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$.

**Proof.**

Solution Use the Triangle inequality.

\[
|x^5 - 2x - 5| = |x^5 + (-2x - 5)| \\
\leq |x^5| + |-2x - 5| \\
= |x^5| + |2x + 5| \quad \text{By Triangle ineq} \\
\]

$|y| = | - y| \quad \text{since}$ $|x| \leq 2$ \quad \text{So take $M = 41$.}$
Example

Find an $M$ so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$.

Proof.

Solution Use the Triangle inequality.

\[
\begin{align*}
|x^5 - 2x - 5| &= |x^5 + (-2x - 5)| \\
&\leq |x^5| + |-2x - 5| \\
&= |x^5| + |2x + 5| \\
&\leq |x^5| + |2x| + |5| \\
&\leq 2|5| + 2|2| + 5 = 41 \quad \text{By Triangle ineq}
\end{align*}
\]

So take $M = 41$. 

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Bounding argument

Example

Find an $M$ so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$.

Proof.

Solution Use the Triangle inequality.

$$|x^5 - 2x - 5| = |x^5 + (-2x - 5)|$$
$$\leq |x^5| + |-2x - 5|$$
$$= |x^5| + |2x + 5|$$
$$\leq |x^5| + |2x| + |5|$$
$$= |x|^2 + 2|x| + 5$$

By Triangle ineq

$|y| = |-y|$
Bounding argument

**Example**

Find an $M$ so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$.

**Proof.**

Solution Use the Triangle inequality.

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|x^5 - 2x - 5| = |x^5 + (-2x - 5)| \\
\leq |x^5| + |-2x - 5| \quad \text{By Triangle ineq} \\
= |x^5| + |2x + 5| \\
\leq |x^5| + |2x| + |5| \quad \text{By Triangle ineq} \\
= |x|^2 + 2|x| + 5 \\
\leq 2^5 + 2 \cdot 2 + 5 = 41 \quad \text{since } |x| \leq 2
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Example

Find an $M$ so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$.

Proof.

Solution Use the Triangle inequality.

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|x^5 - 2x - 5| = |x^5 + (-2x - 5)| \\
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\leq 2^5 + 2 \cdot 2 + 5 = 41 \quad \text{since } |x| \leq 2
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So take $M = 41$. 

By Triangle ineq
What’s the value in understanding the absolute value function geometrically as well as algebraically?

How is the triangle inequality related to triangles?

How come Desmos tells us that $|x^5 - 2x - 5| \leq 33$ on the interval $-2 \leq x \leq 2$, but we got an answer of 41?