Introduction to Proofs - Triangle Inequality

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Apply the absolute value function in various contexts (algebraically and geometrically).
2. Relate/distinguish problem solving and rough work to a proof.
3. Break up a sum using the triangle inequality.

Motivation

"The sum of any two sides of a triangle is always greater or equal to the third side."

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The triangle inequality is one of the two important inequalities we see in this course.
We will understand it geometrically and algebraically.

Absolute value

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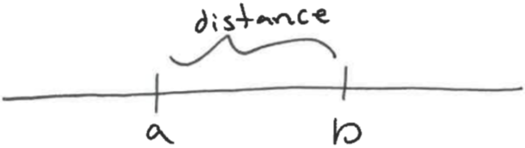
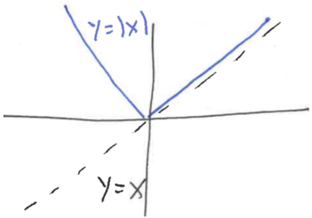
Special case. $|a - b|$ is the distance between a and b.



More visualizations of absolute value

This picture shows us the inequality: for all real x, $x \leq |x|$.



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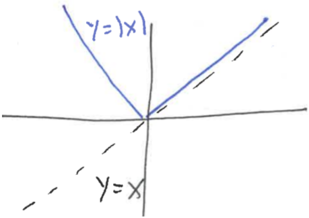

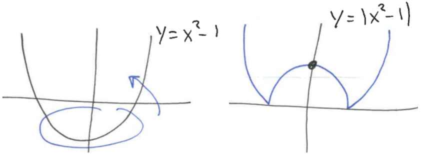

What is the max value of $|x^2 - 1|$ when $|x| \leq 1$?

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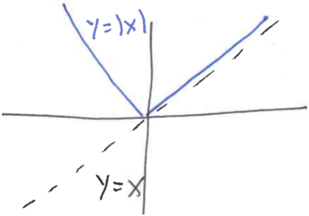

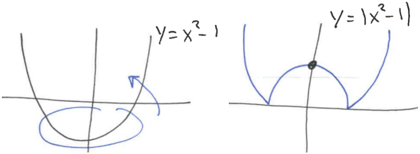

What is the max value of $|x^2 - 1|$ when $|x| \leq 1$? At $x = 0$, $|x^2 - 1| = 1$.

Basic Facts about absolute value

Facts

Let $x, y \in \mathbb{R}$.

1. $x^2 = |x|^2$
2. $\sqrt{x^2} = |x|$
3. $x \leq |x|$
4. $|xy| = |x||y|$

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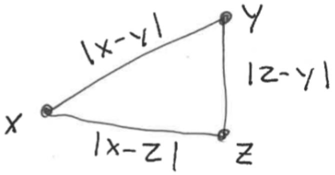
Special case of this. $|b - a| = |-(a - b)| = |a - b|$.
The distance from a to b is the same as the distance from b to a.

Motivation for Triangle Inequality

$$
\begin{aligned}
|x - y| &= |(x - z) + (z - y)| \\
&\leq |x - z| + |z - y|
\end{aligned}
$$

Intuition. Direct routes are shorter than detours (to z).



Triangle Inequality - Rough work

Theorem (Triangle Inequality)

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

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&|x + y| \leq |x| + |y| \\
&\implies |x + y|^2 \leq (|x| + |y|)^2 \\
&\implies (x + y)^2 \leq |x|^2 + 2|x||y| + |y|^2
\end{aligned}
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&\implies x^2 + 2xy + y^2 \leq |x|^2 + 2|xy| + |y|^2
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&\implies 2xy \leq 2|xy| \\
&\implies xy \leq |xy| \longleftarrow \text{ This is the basic fact we will start from.}
\end{aligned}$$

Proof of Triangle inequality

Proof.

Let $x, y \in \mathbb{R}$.
Note that $xy \leq |xy|$ by Fact 3.
So $2xy \leq 2|xy| = 2|x||y|$.
Since, by Fact 1, $x^2 = |x|^2$ and $y^2 = |y|^2$ we get

$$x^2 + 2xy + y^2 \leq |x|^2 + 2|xy| + |y|^2.$$

Factoring gives $|x+y|^2 \leq (|x|+|y|)^2$. By Fact 2 of inequalities,
$|x+y| \leq ||x|+|y||$.
Note that $||x|+|y|| = |x|+|y|$, since $|x| \geq 0, |y| \geq 0$ and $|x|+|y| \geq 0$.
So $|x+y| \leq |x|+|y|$. □

Bounding argument

Example

Find an M so that $|x^5 - 2x - 5| \leq M$ for all $|x| \leq 2$.

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Solution Use the Triangle inequality.

$$|x^5 - 2x - 5| = |x^5 + (-2x - 5)|$$

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$\square$

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&= |x|^2 + 2|x| + 5 & \\
&\leq 2^5 + 2 \cdot 2 + 5 = 41 & \text{since } |x| \leq 2
\end{aligned}$$

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Bounding argument

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&= |x|^2 + 2|x| + 5 && \\
&\leq 2^5 + 2 \cdot 2 + 5 = 41 && \text{since } |x| \leq 2
\end{aligned}$$

So take $M = 41$. □

Reflection

- What's the value in understanding the absolute value function geometrically as well as algebraically?
- How is the triangle inequality related to triangles?
- How come Desmos tells us that $|x^5 - 2x - 5| \leq 33$ on the interval $-2 \leq x \leq 2$, but we got an answer of 41?