Introduction to Proofs - Functions (range calculations)

Prof Mike Pawliuk

UTM

July 7, 2020

Slides available at: mikepawliuk.ca

Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Compute the range of a rational function $f : \mathbb{R} \to \mathbb{R}$, and prove that it is correct.

Motivation

Motivation

Now that we are able to manipulate inequalities, we return to functions to compute ranges of rational functions. These functions often show up in calculus, but we will make these computations without calculus.

1. Review

Let $f : A \to B$ be a function, and $C \subseteq A$.

- $\text{dom}(f) = A$,
- $\text{codom}(f) = B$,
- $\text{ran}(f) = \{f(a) : a \in A\}$,
- Image of C under f is $\{f(c) : c \in C\}$.

1. Review

Let $f : A \to B$ be a function, and $C \subseteq A$.

- $\text{dom}(f) = A$,
- $\text{codom}(f) = B$,
- $\text{ran}(f) = \{f(a) : a \in A\}$,
- Image of C under f is $\{f(c) : c \in C\}$.

Example

Let $f : \{1, 2, 3\} \to \mathbb{R}$ be defined by $f(1) = 1, f(2) = 2$ and $f(3) = 1$. and let $C = \{1, 3\}$.

1. Review

Let $f : A \to B$ be a function, and $C \subseteq A$.

- $\text{dom}(f) = A$,
- $\text{codom}(f) = B$,
- $\text{ran}(f) = \{f(a) : a \in A\}$,
- Image of C under f is $\{f(c) : c \in C\}$.

Example

Let $f : \{1, 2, 3\} \to \mathbb{R}$ be defined by $f(1) = 1, f(2) = 2$ and $f(3) = 1$. and let $C = \{1, 3\}$.

- $\text{dom}(f) = \{1, 2, 3\}$,
- $\text{codom}(f) = \mathbb{R}$,
- $\text{ran}(f) = \{f(a) : a \in A\} = \{f(1), f(2), f(3)\} = \{1, 2\}$,
- Image of C under f is $\{f(c) : c \in C\} = \{f(1), f(3)\} = \{1\}$.

2. Warm up example

Warm up

Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = |x - 1| + 1$.
Prove that $\text{ran}(f) = \mathbb{N}$.

2. Warm up example

Warm up

Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = |x - 1| + 1$.
Prove that $\text{ran}(f) = \mathbb{N}$.

Ursula's claim: "Absolute values are always non-negative, and $|x - 1| + 1$ is an integer if x is, so $f(x)$ is a natural number."

2. Warm up example

Warm up

Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = |x - 1| + 1$.
Prove that $\text{ran}(f) = \mathbb{N}$.

Ursula's claim: "Absolute values are always non-negative, and $|x - 1| + 1$ is an integer if x is, so $f(x)$ is a natural number."

Is this a complete proof?

2. Warm up example

Warm up

Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = |x - 1| + 1$.
Prove that $\text{ran}(f) = \mathbb{N}$.

Ursula's claim: "Absolute values are always non-negative, and $|x - 1| + 1$ is an integer if x is, so $f(x)$ is a natural number."

Is this a complete proof?

No. It only shows that $\text{ran}(f) \subseteq \mathbb{N}$.

2. Warm up example

Warm up

Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = |x-1|+1$.
Prove that $\operatorname{ran}(f) = \mathbb{N}$.

Ursula's claim: "Absolute values are always non-negative, and $|x-1|+1$ is an integer if x is, so $f(x)$ is a natural number."

Is this a complete proof?

No. It only shows that $\operatorname{ran}(f) \subseteq \mathbb{N}$.

How do we know that $\mathbb{N} \subseteq$ ran(f)? In other words, how do we know that every $n \in \mathbb{N}$ can be written as $|x-1|+1$ for some $x \in \mathbb{Z}$?

continued

Warm up

Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = |x - 1| + 1$.
Prove that $\text{ran}(f) = \mathbb{N}$.

Suppose $n \in \mathbb{N}$. Let $x = -(n - 1) + 1$.

continued

Warm up

Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = |x - 1| + 1$.
Prove that $\text{ran}(f) = \mathbb{N}$.

Suppose $n \in \mathbb{N}$. Let $x = -(n-1) + 1$.

1. Note that $x \in \mathbb{Z} = \text{dom}(f)$.
2. Note that
   $f(x) = |-(n-1)+1-1| + 1 =$

continued

Warm up

Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = |x - 1| + 1$.
Prove that $\text{ran}(f) = \mathbb{N}$.

Suppose $n \in \mathbb{N}$. Let $x = -(n-1) + 1$.

1. Note that $x \in \mathbb{Z} = \text{dom}(f)$.
2. Note that
   $f(x) = |-(n-1) + 1 - 1| + 1 = |-(n-1)| + 1 = n - 1 + 1 = n.$

So $n \in \text{ran}(f)$.

Proof strategy

Proof strategy for showing $C = \text{ran}(f)$.

To show that $C = \text{ran}(f)$ you need to show double subset.

1. $\text{ran}(f) \subseteq C$. This means showing that every $f(x)$ is in C.
2. $C \subseteq \text{ran}(f)$. This means that every $c \in C$ can be written as $f(x) = c$. It is your job to find one x that works.

3. An example with a rational function

Example

Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{x}{1+x^2}$. Show $\operatorname{ran}(f) = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

3. An example with a rational function

Example

Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{x}{1+x^2}$. Show $\text{ran}(f) = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

$\boxed{\text{ran}(f) \subseteq \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]}$ Let $y \in \text{ran}(f)$. So there is an $x \in \mathbb{R}$ such that $f(x) = y$.

Note

$\implies$

$\implies$

$\implies$

$\implies$

3. An example with a rational function

Example

Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{x}{1+x^2}$. Show $\text{ran}(f) = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

$\boxed{\text{ran}(f) \subseteq \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]}$ Let $y \in \text{ran}(f)$. So there is an $x \in \mathbb{R}$ such that $f(x) = y$.

$$\text{Note } -(1+x)^2 \leq 0 \leq (1-x)^2$$

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

Example

Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{x}{1+x^2}$. Show $\text{ran}(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

$\boxed{\text{ran}(f) \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]}$ Let $y \in \text{ran}(f)$. So there is an $x \in \mathbb{R}$ such that $f(x) = y$.

$$\begin{aligned} \text{Note } & -(1+x)^2 \leq 0 \leq (1-x)^2 \\ \implies & -1 - x^2 - 2x \leq 0 \leq 1 - 2x + x^2 \\ \implies & \\ \implies & \\ \implies & \end{aligned}$$

3. An example with a rational function

Example

Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{x}{1+x^2}$. Show $\text{ran}(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

$\boxed{\text{ran}(f) \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]}$ Let $y \in \text{ran}(f)$. So there is an $x \in \mathbb{R}$ such that $f(x) = y$.

$$\begin{aligned} \text{Note } & -(1+x)^2 \leq 0 \leq (1-x)^2 \\ \implies & -1 - x^2 - 2x \leq 0 \leq 1 - 2x + x^2 \\ \implies & -(1+x^2) \leq 2x \leq 1 + x^2 \\ \implies & \\ \implies & \end{aligned}$$

3. An example with a rational function

Example

Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{x}{1+x^2}$. Show $\text{ran}(f) = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

$\boxed{\text{ran}(f) \subseteq \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]}$ Let $y \in \text{ran}(f)$. So there is an $x \in \mathbb{R}$ such that $f(x) = y$.

$$\begin{aligned}
\text{Note } & -(1+x)^2 \le 0 \le (1-x)^2 \\
\implies & -1 - x^2 - 2x \le 0 \le 1 - 2x + x^2 \\
\implies & -(1+x^2) \le 2x \le 1 + x^2 \\
\implies & -1 \le \frac{2x}{1+x^2} \le 1 \qquad \text{since } 1/(1+x^2) > 0 \\
\implies &
\end{aligned}$$

3. An example with a rational function

Example

Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{x}{1+x^2}$. Show $\text{ran}(f) = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

$\boxed{\text{ran}(f) \subseteq \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]}$ Let $y \in \text{ran}(f)$. So there is an $x \in \mathbb{R}$ such that $f(x) = y$.

$$
\begin{aligned}
\text{Note } & -(1+x)^2 \leq 0 \leq (1-x)^2 \\
\implies & -1 - x^2 - 2x \leq 0 \leq 1 - 2x + x^2 \\
\implies & -(1+x^2) \leq 2x \leq 1 + x^2 \\
\implies & -1 \leq \frac{2x}{1+x^2} \leq 1 \qquad \text{since } 1/(1+x^2) > 0 \\
\implies & -\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2}
\end{aligned}
$$

Example continued

$$\implies -\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2}$$

$$\implies$$

$$\implies$$

$$\implies$$

Example continued

$$\implies -\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2}$$
$$\implies -\frac{1}{2} \leq y \leq \frac{1}{2}$$
$$\implies$$
$$\implies$$

Example continued

$$\implies -\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2}$$
$$\implies -\frac{1}{2} \leq y \leq \frac{1}{2}$$
$$\implies y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$
$$\implies$$

Example continued

$$
\begin{aligned}
&\implies -\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2} \\
&\implies -\frac{1}{2} \leq y \leq \frac{1}{2} \\
&\implies y \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\
&\implies y \in \operatorname{ran}(f).
\end{aligned}
$$

3. Example continued

$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \text{ran}(f)}$ Let $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We need to find an $x \in \mathbb{R}$ such that $f(x) = y$.

3. Example continued

$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \text{ran}(f)}$ Let $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We need to find an $x \in \mathbb{R}$ such that $f(x) = y$.

<u>Case 1</u>. If $y = 0$, then let $x = 0$. Note $f(0) = 0 = y$.

3. Example continued

$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \text{ran}(f)}$ Let $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We need to find an $x \in \mathbb{R}$ such that $f(x) = y$.

Case 1. If $y = 0$, then let $x = 0$. Note $f(0) = 0 = y$.

Case 2. Suppose $y \neq 0$. Let $x = \dfrac{1 + \sqrt{1 - 4y^2}}{2y}$. (**Exercise**. Show this is always a real number.)

3. Example continued

$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \text{ran}(f)}$ Let $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We need to find an $x \in \mathbb{R}$ such that $f(x) = y$.

Case 1. If $y = 0$, then let $x = 0$. Note $f(0) = 0 = y$.

Case 2. Suppose $y \neq 0$. Let $x = \dfrac{1 + \sqrt{1 - 4y^2}}{2y}$. (**Exercise**. Show this is always a real number.)

Note that this x is a solution to the following equation (by the quadratic formula):

$$yx^2 - x + y = 0$$

$$\Longrightarrow$$

$$\Longrightarrow \qquad .$$

$$\Longrightarrow$$

3. Example continued

$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \text{ran}(f)}$ Let $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We need to find an $x \in \mathbb{R}$ such that $f(x) = y$.

Case 1. If $y = 0$, then let $x = 0$. Note $f(0) = 0 = y$.

Case 2. Suppose $y \neq 0$. Let $x = \dfrac{1 + \sqrt{1 - 4y^2}}{2y}$. (**Exercise**. Show this is always a real number.)

Note that this x is a solution to the following equation (by the quadratic formula):

$$
\begin{aligned}
& yx^2 - x + y = 0 \\
\implies & y(x^2 + 1) = x \\
\implies & \qquad\qquad\qquad . \\
\implies &
\end{aligned}
$$

3. Example continued

$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \text{ran}(f)}$ Let $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We need to find an $x \in \mathbb{R}$ such that $f(x) = y$.

Case 1. If $y = 0$, then let $x = 0$. Note $f(0) = 0 = y$.

Case 2. Suppose $y \neq 0$. Let $x = \dfrac{1 + \sqrt{1 - 4y^2}}{2y}$. (**Exercise**. Show this is always a real number.)

Note that this x is a solution to the following equation (by the quadratic formula):

$$\begin{aligned} & yx^2 - x + y = 0 \\ \implies & y(x^2 + 1) = x \\ \implies & y = \frac{x}{1 + x^2} = f(x). \\ \implies & \end{aligned}$$

3. Example continued

$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \text{ran}(f)}$ Let $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We need to find an $x \in \mathbb{R}$ such that $f(x) = y$.

Case 1. If $y = 0$, then let $x = 0$. Note $f(0) = 0 = y$.

Case 2. Suppose $y \neq 0$. Let $x = \dfrac{1 + \sqrt{1 - 4y^2}}{2y}$. (**Exercise**. Show this is always a real number.)

Note that this x is a solution to the following equation (by the quadratic formula):

$$\begin{aligned} & yx^2 - x + y = 0 \\ \implies & y(x^2 + 1) = x \\ \implies & y = \frac{x}{1 + x^2} = f(x). \\ \implies & y \in \text{ran}(f). \end{aligned}$$

Reflection

- What is the difference between showing $\text{ran}(f) \subseteq C$ and showing $C \subseteq \text{ran}(f)$.
- What role did the quadratic equation play in the previous example?
- How would you have written up your explanation of the previous example?