Introduction to Proofs - Sum and Product notation

Prof Mike Pawliuk

UTM

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Slides available at: mikepawliuk.ca

Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Write a sum/product in sigma/pi notation.
2. Extract a sum/product from the sigma/pi notation.

Motivation

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We want to express things like

$$1 + 2 + 3 + \ldots + 100$$

in a more compact and precise way, without using "...".
This will be especially useful in calculus and stats (e.g. Riemann sums).

1. Warm-up example

1. Compute the following sums.
2. Make a general conjecture.
3. Express your conjecture in precise mathematical language.

$$
\begin{aligned}
&= 1 \\
&= 1 + 3 \\
&= 1 + 3 + 5 \\
&= 1 + 3 + 5 + 7 \\
&= 1 + 3 + 5 + 7 + 9
\end{aligned}
$$

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9 &= 1 + 3 + 5 \\
16 &= 1 + 3 + 5 + 7 \\
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Conjecture: The sum of consecutive odd numbers is a square.

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Conjecture: The sum of consecutive odd numbers is a square.
Conjecture: $(\forall n \in \mathbb{N})[1+3+\ldots+(2n-3)+(2n-1)=n^2]$

2. Making this precise: The math way

Definition (Sigma notation)

Let a_i be a real number (for $i \in \mathbb{N}$) and let $n \in \mathbb{N}$.

$$\sum_{i=1}^{n} a_i =$$

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3. Making this precise: The Python way

This is a for/while loop that computes $\sum_{i=1}^{n}(2i-1)$

```
def total(n):
    """Give the sum of the odd numbers a_1 through a_n."""
    sum = 0
    for i in range(1, n+1):
        sum += 2*i - 1
    return sum

total(5)
>>>> 25
```

4. Examples

$$\sum_{i=1}^{3} i^2 =$$

$$\sum_{k=1}^{3} k =$$

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$$\sum_{i=-2}^{3} i^2 = (-2)^2 + (-1)^2 + (0)^2 + (1)^1 + (2)^2 + (3)^2 = 19$$

5. Sum exercises

Compute

$$\sum_{i=2}^{3} i =$$

$$\sum_{k=1}^{4} 2 =$$

$$\sum_{j=1}^{2} \left(\sum_{i=1}^{j} i + j \right) =$$

$$=$$

Compute

$$\sum_{i=2}^{3} i = 2 + 3 = 5$$

$$\sum_{k=1}^{4} 2 =$$

$$\sum_{j=1}^{2} \left(\sum_{i=1}^{j} i + j \right) =$$

$$=$$

Compute

$$\sum_{i=2}^{3} i = 2 + 3 = 5$$

$$\sum_{k=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$$

$$\sum_{j=1}^{2} \left(\sum_{i=1}^{j} i + j \right) =$$

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$$\sum_{j=1}^{2} \left(\sum_{i=1}^{j} i + j \right) = \left(\sum_{i=1}^{1} i + 1 \right) + \left(\sum_{i=1}^{2} i + 2 \right)$$

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$$= (1 + 1) + ((1 + 2) + (2 + 2)) = 9$$

6. Sum Theorems

Let c be a real number, and n be a natural number.

$$\sum_{i=1}^{n} 1 =$$

$$\sum_{i=1}^{n} ca_i =$$

$$\sum_{i=1}^{n} (a_i + b_i) =$$

Exercise. If $m \leq n$ then

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Exercise. If $m \leq n$ then

$$\sum_{i=m}^{n} 1 = n - m$$

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Exercise. If $m \leq n$ then

$$\sum_{i=m}^{n} 1 = n - m + 1$$

7. Product notation

Definition (Pi notation)

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Example.

$$\prod_{i=1}^{4} i = 1 \cdot 2 \cdot 3 \cdot 4 = 4!$$

8. Brain melting

Exercise. Write the for loop that produce product.

Convention

$$\sum_{i=1}^{0} a_i = \qquad \text{(empty sum)}$$

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This is one reason why $0! = 1$ and $x^0 = 1$.

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$$\prod_{i=1}^{n} (a_i \cdot b_i) = \left(\prod_{i=1}^{n} a_i \right) \cdot \left(\prod_{i=1}^{n} b_i \right)$$

Compute

$$\sum_{i=1}^{3}\left(\frac{\sum_{j=1}^{i} j}{\prod_{j=1}^{i} j}\right)$$

Reflection

- How do you recognize the dummy variable, the starting index, the end index and the terms you are adding?
- What are the similarities and differences between sum notation and product notation?
- How is sum notation like a for loop in programming?