Introduction to Proofs - Induction - Intro

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Identify a type of statement that could be proved by induction.
2. Produce the structure of a proof by induction.

Motivation

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Induction is a powerful proof technique that allows us to prove results about objects with self-similarity and symmetry.
We will see many variations on it in this course.

1. Warm-up

Assume that

$$1 + 3 + 5 + 7 + 9 + 11 + 13 = 7^2.$$

Then what is

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Then what is

$$(1 + 3 + 5 + 7 + 9 + 11 + 13) + 15 = 7^2 + 15 = 64 = 8^2$$

Observation. We can get from one equation to the next without re-computing everything.

2. Motivating question

Question. What is $1 + 3 + 5 + \ldots + 99$?

$$1 = 1^2$$

For a natural number n, let $P(n)$ be the statement:

$$1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1) = n^2.$$

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$$1 = 1^2 \Rightarrow 2^2 = 4 = 1 + 3$$

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$$\begin{aligned} 1 = 1^2 &\Rightarrow 2^2 = 4 = 1 + 3 \\ &\Rightarrow 3^2 = 9 = 1 + 3 + 5 \\ &\Rightarrow 4^2 = 16 = 1 + 3 + 5 + 7 \end{aligned}$$

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$$\begin{aligned} 1 = 1^2 &\Rightarrow 2^2 = 4 = 1 + 3 \\ &\Rightarrow 3^2 = 9 = 1 + 3 + 5 \\ &\Rightarrow 4^2 = 16 = 1 + 3 + 5 + 7 \\ &\ldots \\ &\Rightarrow 50^2 = 1 + 3 + 5 + \ldots 99 \end{aligned}$$

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We showed: $P(1), P(2), P(3)$ and $P(4)$ are true. Also, the warm-up was $P(7) \implies P(8)$.

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Idea of induction

1. Show that $P(1)$ is true.
2. Show that $P(n) \implies P(n+1)$, for <u>every</u> $n \in \mathbb{N}$.

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Important Note. Part 2 is not saying: $(\forall n \in \mathbb{N})[P(n)]$. It's saying "you can always move up one step".

2. Motivating Example

For a natural number n, let $P(n)$ be the statement:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

Theorem

$\forall n \in \mathbb{N}(P(n) \implies P(n+1)).$

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Let $n \in \mathbb{N}$. Assume $P(n)$ is true <u>for this particular n</u>.

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&= n^2 + (2n + 1) \\
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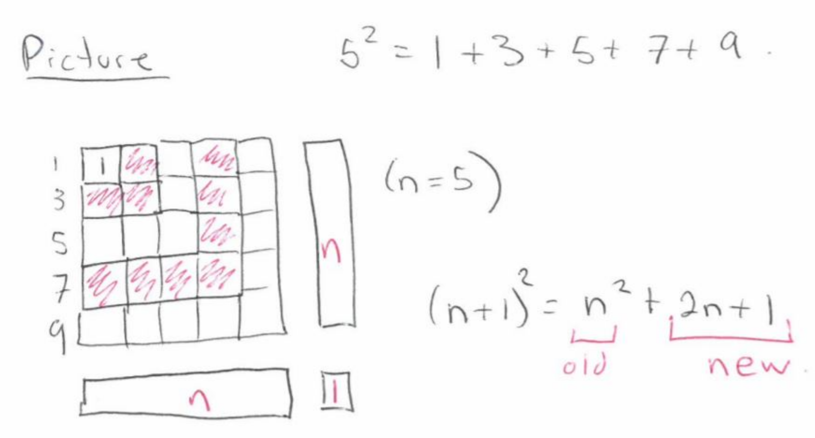
$$\begin{aligned} & 1 + 3 + 5 + \ldots + (2n-1) + (2(n+1) - 1) \\ &= 1 + 3 + 5 + \ldots + (2n-1) + (2n + 2 - 1) \\ &= n^2 + (2n+1) \\ &= (n+1)^2 \qquad \text{So } P(n+1). \end{aligned}$$

2. Motivating Example

Here's a picture of what we just did:



Proof strategy (Regular induction)

To prove a statement of the form "$(\forall n \in \mathbb{N})[P(n)]$"
You can show that:

1. $P(1)$ is true, and
2. $\forall n \in \mathbb{N},\ P(n) \implies P(n+1)$.

3. Mathematical Induction

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You can show that:

1. $P(1)$ is true, and
2. $\forall n \in \mathbb{N},\ P(n) \implies P(n+1)$.

1. $P(1)$ is called the base case.
2. $P(n) \implies P(n+1)$ is called the induction step.
3. $P(n)$ is called the induction hypothesis (IH).

5. Example

Theorem

$\forall n \in \mathbb{N}, 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

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Proof.

Let $P(n)$ be the statement $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. We proceed by induction.

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Theorem

$\forall n \in \mathbb{N}, 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Proof.

Let $P(n)$ be the statement $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. We proceed by induction.

$\boxed{\text{Base case}}$. Note that $\frac{1 \cdot (1+1)}{2} = \frac{2}{2} = 1$. So $P(1)$.

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Induction step Let $n \in \mathbb{N}$. Assume $P(n)$ is true for this particular n.

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Want $P(n+1)$ to be true.

$$1 + 2 + 3 + \ldots + n + (n+1)$$

$=$ by the IH

$=$

$=$

$=$

$\square$

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Want $P(n+1)$ to be true.

$$
\begin{aligned}
& 1 + 2 + 3 + \ldots + n + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{by the IH} \\
&= \\
&= \\
&=
\end{aligned}
$$

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Proof.

Induction step: Let $n \in \mathbb{N}$. Assume $P(n)$ is true for this particular n.
So $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for this particular n.
Want $P(n+1)$ to be true.

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&1 + 2 + 3 + \ldots + n + (n+1) \\
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&= \frac{(n+1)(n+2)}{2} \qquad \text{So } P(n+1).
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5. Example, examined

Let $P(n)$ be the statement $1+2+3+\ldots+n=\frac{n(n+1)}{2}$. We proceed by induction.

$\boxed{\text{Base case}}$. Note that $\frac{1\cdot(1+1)}{2}=\frac{2}{2}=1$. So $P(1)$.

$\boxed{\text{Induction step}}$ Let $n \in \mathbb{N}$. Assume $P(n)$ is true <u>for this particular n</u>.

So $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ <u>for this particular n</u>.

Want $P(n+1)$ to be true.

$$
\begin{aligned}
&1+2+3+\ldots+n+(n+1) \\
&=\frac{n(n+1)}{2}+(n+1) \qquad \text{by the IH} \\
&=\frac{n(n+1)}{2}+\frac{2(n+1)}{2} \\
&=\frac{n(n+1)+2(n+1)}{2} \\
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Let $P(n)$ be the statement $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. We proceed by induction.

Base case. Note that $\frac{1 \cdot (1+1)}{2} = \frac{2}{2} = 1$. So $P(1)$.

Induction step Let $n \in \mathbb{N}$. Assume $P(n)$ is true

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Reflection

- How is induction like knocking over a stack of dominoes?
- What are the essential features of induction?
- Why did we emphasize that the inductive hypothesis is "$P(n)$, for one particular n"?