Introduction to Proofs - Induction - Variations

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Prove a statement by induction, starting somewhere other than $n = 1$.
2. Prove a statement by induction on the evens or odds only.

Motivation

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Induction has many variations. We can change the starting value, and how far the "steps" are.
Today we will see two of those variations.

1. Induction can start at different base cases

Question: How do n^2 and 2^n compare?

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1			
2			
3			
4			
5			
6			

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3	9	8	No
4	16	16	Yes
5	25	32	Yes
6	36	64	Yes

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$$(n+1)^2 = n^2 + (2n+1)$$

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$$\begin{aligned}(n+1)^2 &= n^2 + (2n+1) \\ &\leq 2^n + 2^n \qquad \text{by IH and lemma below}\end{aligned}$$

□

<u>Lemma</u>. For all $n \geq 4$ (and $n \in \mathbb{N}$) we have $2n + 1 \leq 2^n$.

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Lemma. For all $n \geq 4$ (and $n \in \mathbb{N}$) we have $2n + 1 \leq 2^n$.

1. Induction can start at different base cases

Proof strategy (Induction at other base cases)

Let $N \in \mathbb{N}$. To prove "$(\forall n \geq N)P(n)$". (Here $n \in \mathbb{N}$.)

1. Prove $P(N)$, and
2. Show if $n \geq N$, then $P(n) \implies P(n+1)$.

2. Induction can have different "jumps"

Proof strategy (Induction on evens)

Let $N \in \mathbb{N}$. To prove "for all even natural n, P(n)".

1. Prove $P(2)$, and
2. Show $P(n) \implies P(n+2)$ for all even $n \geq 2$.

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Proof strategy (Induction on evens)

Let $N \in \mathbb{N}$. To prove "for all even natural n, P(n)".

1. Prove $P(2)$, and
2. Show $P(n) \implies P(n+2)$ for all even $n \geq 2$.

There is a version for odds as well.

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Theorem

For every even natural n, $n(n^2 + 3n + 2)$ is divisible by 24.

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Proof.

By induction (on the evens). Let $P(n)$ be "$n(n^2 + 3n + 2)$ is a multiple of 24".

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For every even natural n, $n(n^2 + 3n + 2)$ is divisible by 24.

Proof.

By induction (on the evens). Let $P(n)$ be "$n(n^2 + 3n + 2)$ is a multiple of 24".

$\boxed{P(2)}$ Note that

$$2(2^2 + 3(2) + 2) = 2(4 + 6 + 2) = 2(12) = 24,$$

which is divisible by 24.

□

Proof continued

Proof.

Let $n \in \mathbb{N}$ be even and assume $P(n)$.
Idea: Find $n(n^2 + 3n + 2)$ somewhere.

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$$(n+2)((n+2)^2 + 3(n+2) + 2)$$

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Proof continued

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Let $n \in \mathbb{N}$ be even and assume $P(n)$.
<u>Idea</u>: Find $n(n^2 + 3n + 2)$ somewhere. Note:

$$
\begin{aligned}
&(n+2)((n+2)^2 + 3(n+2) + 2) \\
&= (\underline{n} + 2)(\underline{n^2} + 4n + 4 + \underline{3n} + 6 + \underline{2})
\end{aligned}
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&= n\left([n^2 + 3n + 2] + [4n + 4 + 6]\right) + 2(n^2 + 7n + 12)
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First term is divisible by 24 by IH.

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First term is divisible by 24 by IH.
Second term is because n is even.

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First term is divisible by 24 by IH.
Second term is because n is even.
Third term is obvious.
So the sum is divisible by 24.

□

3. Other gaps

Example

Suppose someone gives you a function $f : \mathbb{Z} \to \{0, 1\}$ and tells you:

1. For all $x \in \mathbb{Z}$, if $f(x) = 1$, then $f(x + 3) = 1$.
2. For all $x \in \mathbb{Z}$, if $f(x) = 1$, then $f(x + 5) = 1$.
3. $f(0) = 1$.

What other numbers $x \in \mathbb{Z}$ can you conclude must have $f(x) = 1$?

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$0, 3, 6, 9, 12, 15, \ldots$
$0, 5, 10, 15, 20, \ldots$

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$0, 3, 5, 6, 8, 9, 10, [11, 12, 13], \ldots$

4. Exercises

1. Find, with proof, all values of n such that $2^n \leq n!$.
2. You have a pencil that is 20 cm long, and a pencil sharpener that can take off either 3cm or 5cm at a time. What possible lengths can you make your pencil?
3. Show that for all $k \in \mathbb{N}$, there is an N such that $n^k \leq 2^n$ for all $n \in \mathbb{N}$ with $n \geq N$.

Reflection

- What are the two things you can modify about simple induction?
- Is there a variation on induction to prove something about all negative integers?
- Is there a variation on induction to prove something about all positive multiples of 5?