Introduction to Proofs - Induction - Recursion

Prof Mike Pawliuk

UTM

July 16, 2020

Slides available at: mikepawliuk.ca

Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Distinguish between explicit and recursive descriptions of sequences.
2. Prove a statement about a recursively defined sequence using induction.

Motivation

Motivation

We want to describe a sequence of real numbers $a_1, a_2, a_3, \ldots,$ "locally".

Motivation

Motivation

We want to describe a sequence of real numbers $a_1, a_2, a_3, \ldots,$ "locally".

Motivation

Induction and recursion are two sides of the same coin.
You use induction to prove statements about recursion.

1. Example

Example

Define $a_1 = 6$ and $a_{n+1} = 5a_n + 1$ for all $n \in \mathbb{N}$.

1. Example

Example

Define $a_1 = 6$ and $a_{n+1} = 5a_n + 1$ for all $n \in \mathbb{N}$.

Note

1. $a_1 = 6$.

1. Example

Example

Define $a_1 = 6$ and $a_{n+1} = 5a_n + 1$ for all $n \in \mathbb{N}$.

Note

1. $a_1 = 6$.
2. $a_2 = a_{1+1} = 5(a_1) + 1 = 5(6) + 1 = 31$

1. Example

Example

Define $a_1 = 6$ and $a_{n+1} = 5a_n + 1$ for all $n \in \mathbb{N}$.

Note

1. $a_1 = 6$.
2. $a_2 = a_{1+1} = 5(a_1) + 1 = 5(6) + 1 = 31$
3. $a_3 = 5(31) + 1 = 156$

1. Example

Example

Define $a_1 = 6$ and $a_{n+1} = 5a_n + 1$ for all $n \in \mathbb{N}$.

Note

1. $a_1 = 6$.
2. $a_2 = a_{1+1} = 5(a_1) + 1 = 5(6) + 1 = 31$
3. $a_3 = 5(31) + 1 = 156$

Compute a_{10} in two ways:

- Recursive $a_1 = 6$, $a_{n+1} = 5a_n + 1$.
- Explicit $a_n = \dfrac{5^{n+1} - 1}{4}$

1. Example

Example

Define $a_1 = 6$ and $a_{n+1} = 5a_n + 1$ for all $n \in \mathbb{N}$.

Note

1. $a_1 = 6$.
2. $a_2 = a_{1+1} = 5(a_1) + 1 = 5(6) + 1 = 31$
3. $a_3 = 5(31) + 1 = 156$

Compute a_{10} in two ways:

- $\boxed{\text{Recursive}}$ $a_1 = 6,\ a_{n+1} = 5a_n + 1.$ Approx 90 seconds.
- $\boxed{\text{Explicit}}$ $a_n = \dfrac{5^{n+1} - 1}{4}$ Approx 10 seconds.

2. Proof about recursive sequences

$a_1 = 6,\ a_{n+1} = 5a_n + 1$

Theorem

For the sequence a_n defined previously, $\forall n \in \mathbb{N}$ we have $a_n = \dfrac{5^{n+1} - 1}{4}$.

2. Proof about recursive sequences

$a_1 = 6, \ a_{n+1} = 5a_n + 1$

Theorem

For the sequence a_n defined previously, $\forall n \in \mathbb{N}$ we have $a_n = \dfrac{5^{n+1} - 1}{4}$.

Proof.

By induction.

2. Proof about recursive sequences

$a_1 = 6, \ a_{n+1} = 5a_n + 1$

Theorem

For the sequence a_n defined previously, $\forall n \in \mathbb{N}$ we have $a_n = \dfrac{5^{n+1} - 1}{4}$.

Proof.

By induction.

Base Note

$$\frac{5^{1+1} - 1}{4} = \frac{25 - 1}{4} = 6 = a_1.$$

□

2. Proof about recursive sequences

$a_1 = 6,\ a_{n+1} = 5a_n + 1$

Proof.

Induction step. Assume that $a_n = \dfrac{5^{n+1} - 1}{4}$ for a particular $n \in \mathbb{N}$.

2. Proof about recursive sequences

$a_1 = 6,\ a_{n+1} = 5a_n + 1$

Proof.

Induction step. Assume that $a_n = \dfrac{5^{n+1} - 1}{4}$ for a particular $n \in \mathbb{N}$. Note that

$$a_{n+1} =$$

2. Proof about recursive sequences

$a_1 = 6,\ a_{n+1} = 5a_n + 1$

Proof.

Induction step. Assume that $a_n = \dfrac{5^{n+1} - 1}{4}$ for a particular $n \in \mathbb{N}$.
Note that

$$a_{n+1} = 5a_n + 1 \qquad \text{by definition of } a_n$$

2. Proof about recursive sequences

$a_1 = 6,\ a_{n+1} = 5a_n + 1$

Proof.

Induction step. Assume that $a_n = \dfrac{5^{n+1} - 1}{4}$ for a particular $n \in \mathbb{N}$.
Note that

$$\begin{aligned} a_{n+1} &= 5a_n + 1 && \text{by definition of } a_n \\ &= 5\left(\frac{5^{n+1} - 1}{4} + 1\right) && \text{by IH} \end{aligned}$$

2. Proof about recursive sequences

$a_1 = 6, \ a_{n+1} = 5a_n + 1$

Proof.

Induction step. Assume that $a_n = \dfrac{5^{n+1} - 1}{4}$ for a particular $n \in \mathbb{N}$.
Note that

$$\begin{aligned} a_{n+1} &= 5a_n + 1 && \text{by definition of } a_n \\ &= 5\left(\frac{5^{n+1} - 1}{4} + 1\right) && \text{by IH} \\ &= \frac{5^{n+2} - 5}{4} + 1 \end{aligned}$$

2. Proof about recursive sequences

$a_1 = 6,\ a_{n+1} = 5a_n + 1$

Proof.

Induction step. Assume that $a_n = \dfrac{5^{n+1} - 1}{4}$ for a particular $n \in \mathbb{N}$.
Note that

$$\begin{aligned}
a_{n+1} &= 5a_n + 1 && \text{by definition of } a_n \\
&= 5\left(\frac{5^{n+1} - 1}{4} + 1\right) && \text{by IH} \\
&= \frac{5^{n+2} - 5}{4} + 1 \\
&= \frac{5^{n+2} - 5 + 4}{4}
\end{aligned}$$

2. Proof about recursive sequences

$a_1 = 6,\ a_{n+1} = 5a_n + 1$

Proof.

Induction step. Assume that $a_n = \dfrac{5^{n+1}-1}{4}$ for a particular $n \in \mathbb{N}$.
Note that

$$
\begin{aligned}
a_{n+1} &= 5a_n + 1 && \text{by definition of } a_n \\
&= 5\left(\frac{5^{n+1}-1}{4} + 1\right) && \text{by IH} \\
&= \frac{5^{n+2}-5}{4} + 1 \\
&= \frac{5^{n+2}-5+4}{4} \\
&= \frac{5^{n+2}-1}{4}
\end{aligned}
$$

3. Multiple base cases

The Fibonacci numbers (F_n) are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

3. Multiple base cases

The Fibonacci numbers (F_n) are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

defined recursively by

- $F_0 = 0$ and $F_1 = 1$, and
- $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$.

3. Multiple base cases

The Fibonacci numbers (F_n) are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

defined recursively by

- $F_0 = 0$ and $F_1 = 1$, and
- $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$.

The "Fibonacci" numbers were not invented by Fibonacci. He introduced these numbers to Europe in his book *Liber Abaci*.
You can listen to a podcast about this here: `https://n.pr/2Pi9Cr6`

3. Multiple base cases

$F_0 = 0$ and $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$

Theorem

For all integers $n \geq 0$, $F_n \leq 2^n$.

3. Multiple base cases

$F_0 = 0$ and $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$

Theorem

For all integers $n \geq 0$, $F_n \leq 2^n$.

Proof.

We proceed by induction. We skip the base case for now.

Inductive step Assume $F_n \leq 2^n$ for a particular n.

Note

$$F_{n+2} =$$

□

3. Multiple base cases

$F_0 = 0$ and $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$

Theorem

For all integers $n \geq 0$, $F_n \leq 2^n$.

Proof.

We proceed by induction. We skip the base case for now.

Inductive step Assume $F_n \leq 2^n$ for a particular n.

Note

$$F_{n+2} = F_n + F_{n+1} \qquad \text{by definition}$$

□

3. Multiple base cases

$F_0 = 0$ and $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$

Theorem

For all integers $n \geq 0$, $F_n \leq 2^n$.

Proof.

We proceed by induction. We skip the base case for now.

Inductive step Assume $F_n \leq 2^n$ for a particular n. Assume $F_{n+1} \leq 2^{n+1}$.

Note

$$F_{n+2} = F_n + F_{n+1} \qquad \text{by definition}$$

□

3. Multiple base cases

$F_0 = 0$ and $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$

Theorem

For all integers $n \geq 0$, $F_n \leq 2^n$.

Proof.

We proceed by induction. We skip the base case for now.

Inductive step Assume $F_n \leq 2^n$ for a particular n. Assume $F_{n+1} \leq 2^{n+1}$.

Note

$$\begin{aligned} F_{n+2} &= F_n + F_{n+1} && \text{by definition} \\ &\leq 2^n + 2^{n+1} && \text{by IH} \end{aligned}$$

□

3. Multiple base cases

$F_0 = 0$ and $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$

Theorem

For all integers $n \geq 0$, $F_n \leq 2^n$.

Proof.

We proceed by induction. We skip the base case for now.

Inductive step Assume $F_n \leq 2^n$ for a particular n. Assume $F_{n+1} \leq 2^{n+1}$.

Note

$$\begin{aligned} F_{n+2} &= F_n + F_{n+1} && \text{by definition} \\ &\leq 2^n + 2^{n+1} && \text{by IH} \\ &= 2^n(1+2) \leq 2^n(4) \\ &\leq 2^{n+2}. \end{aligned}$$

□

3. Multiple base cases

$$\boxed{F_0 = 0 \text{ and } F_1 = 1,\ F_{n+2} = F_{n+1} + F_n}$$

Proof.

Base case For $n = 0$ we have $F_0 = 0 \leq 1 = 2^0$.
For $n = 1$ we have $F_1 = 1 \leq 2 = 2^1$. □

3. Multiple base cases

$$F_0 = 0 \text{ and } F_1 = 1,\ F_{n+2} = F_{n+1} + F_n$$

Proof.

Base case: For $n = 0$ we have $F_0 = 0 \leq 1 = 2^0$.
For $n = 1$ we have $F_1 = 1 \leq 2 = 2^1$. □

Question: Why did we have two base cases?

3. Multiple base cases

$F_0 = 0$ and $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$

Proof.

Base case For $n = 0$ we have $F_0 = 0 \leq 1 = 2^0$.
For $n = 1$ we have $F_1 = 1 \leq 2 = 2^1$. □

Question: Why did we have two base cases?
Answer: Because in our induction step we made two (IH) assumptions.

4. Exercises

1. Prove that $F_n \leq 2^{n-1}$ for all non-negative integers.
2. Prove that $F_n \leq (1.7)^{n-1}$ for all non-negative integers.
3. Do better than 1.7 above.
4. Find an upper bound for the "Tribonacci" numbers defined by $T_0 = 0$, $T_1 = 0$, $T_2 = 1$, and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all non-negative integers.

Reflection

- What are the differences between a recursively defined sequence and an explicitly defined sequence?
- Define your own recursive sequence and see what happens to it.