Introduction to Proofs - Even and Odd numbers

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Give a precise definition of even integers and odd integers.
2. Prove a simple fact about even and odd integers using a “definition unwinding” proof.

Even numbers

Definition (even integer)

An integer a is <u>even</u> if there is an integer n such that $a = 2n$.

Examples

1. 14 is even, because $14 = 2 \cdot 7$, and 7 is an integer.
2. -10 is even, because $-10 = 2 \cdot (-5)$, and -5 is an integer.
3. 0 is even, because $0 = 2 \cdot 0$, and 0 is an integer.
4. There are an even number of rowers in the picture below.

This image is used with permission from Pixabay.
`https://pixabay.com/photos/row-boat-rowing-boat-row-boat-team-244554/`

Odd numbers

Definition (odd integer)

An integer b is <u>odd</u> if there is an integer m such that $b = 2m + 1$.

Examples

1. 7 is odd, because $7 = 2 \cdot 3 + 1$, and 3 is an integer.
2. -1 is odd, because $-1 = 2 \cdot (-1) + 1$, and -1 is an integer.

Concept check

Question

Is the number 98765 an odd number?

Yes, because $98765 = 2(4932) + 1$, and 4932 is an integer.

Question

What is wrong with the argument "Yes, because 98765 ends in a 5, which means it is odd."

This is not the definition of odd integer that we are using. To use that you would have to prove the fact:

Theorem

An integer b is odd if and only if its final digit is $1, 3, 5, 7$, or 9.

Definition unwinding

Theorem

If a is an even number, and b is an odd number, then $a + b$ is odd.

Proof.

Let a be an even number, and let b be an odd number.

□

Definition unwinding

Theorem

If a is an even number, and b is an odd number, then $a + b$ is odd.

Proof.

Let a be an even number, and let b be an odd number.

So $a + b$ is an odd number □

Definition unwinding

Theorem

If a is an even number, and b is an odd number, then $a + b$ is odd.

Proof.

Let a be an even number, and let b be an odd number.

So $a + b = 2(\qquad) + 1$. Where is an integer

So $a + b$ is an odd number by the definition of odd number. □

Definition unwinding

Theorem

If a is an even number, and b is an odd number, then $a + b$ is odd.

Proof.

Let a be an even number, and let b be an odd number.
By the definition of even number, there is an integer n so that $a = 2n$.

So $a + b = 2(\qquad) + 1$. Where $\qquad$ is an integer

So $a + b$ is an odd number by the definition of odd number. □

Definition unwinding

Theorem

If a is an even number, and b is an odd number, then $a + b$ is odd.

Proof.

Let a be an even number, and let b be an odd number.
By the definition of even number, there is an integer n so that $a = 2n$.
By the definition of odd number, there is an integer m so that $b = 2m + 1$.

So $a + b = 2(\qquad) + 1$. Where $\qquad$ is an integer

So $a + b$ is an odd number by the definition of odd number. □

Definition unwinding

Theorem

If a is an even number, and b is an odd number, then $a + b$ is odd.

Proof.

Let a be an even number, and let b be an odd number.
By the definition of even number, there is an integer n so that $a = 2n$.
By the definition of odd number, there is an integer m so that $b = 2m + 1$.
Notice $a + b = 2n + 2m + 1$.
So $a + b = 2(\qquad) + 1$. Where $\qquad$ is an integer

So $a + b$ is an odd number by the definition of odd number. □

Definition unwinding

Theorem

If a is an even number, and b is an odd number, then $a + b$ is odd.

Proof.

Let a be an even number, and let b be an odd number.
By the definition of even number, there is an integer n so that $a = 2n$.
By the definition of odd number, there is an integer m so that $b = 2m + 1$.
Notice $a + b = 2n + 2m + 1$.
So $a + b = 2(n + m) + 1$. Where $n + m$ is an integer since both n, m are integers.
So $a + b$ is an odd number by the definition of odd number. □

Exercises

Exercise 1

Prove, by definition unwinding, that:

- The sum of any two odd numbers is even.
- The product of any two odd numbers is odd.
- If you add an integer to itself, then the result is even.

Exercise 2

What other facts about even numbers, odd numbers, addition and multiplication do you know? Write down and prove all facts you know.

Concept check 1

What is wrong with this argument?

Claim

3 is an even number because $3 = 2(1.5)$.

Concept check 2

What is missing in this proof?

Theorem

If a is even, then a^2 is even.

Proof.

Note $a^2 = (2n)^2 = 4n^2 = 2(2n^2)$.

□

Concept check 2

What is missing in this proof?

Theorem

If a is even, then a^2 is even.

Proof.

Let a be an even number. So $a = 2n$ for some integer n.

Note $a^2 = (2n)^2 = 4n^2 = 2(2n^2)$.

Since n is an integer, so is $2n^2$. So a^2 is even by definition.

□

Reflection

- How can you tell if a small bag of stones contains an even number of stones without counting them?
- How would you answer the question "Is π even or odd?"
- Can a number be both even and odd at the same time? Why?