Introduction to Proofs - Axioms and Theorems

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Learning Objectives (for this video)

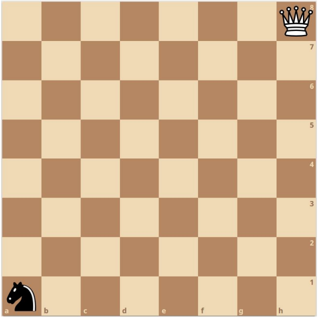
By the end of this video, participants should be able to:

1. Distinguish between an axiom and a proposition/theorem.
2. Make conjectures in axiomatic systems and prove them.

Motivating problem

This is a chess board, with a Knight in the bottom left, and a King in the top right. The king does not move. The Knight can make any number of "L moves" (two-right/one-up, or two-up/one-right) but it must always move up and to the right at least one unit.

What are the possible squares the Knight can reach? In particular, can the Knight reach the King's square?



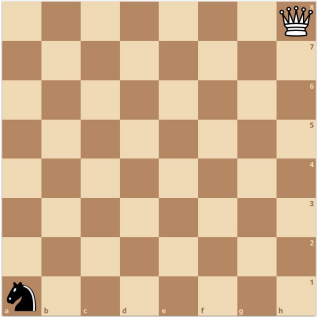
This picture generated from `lichess.org`

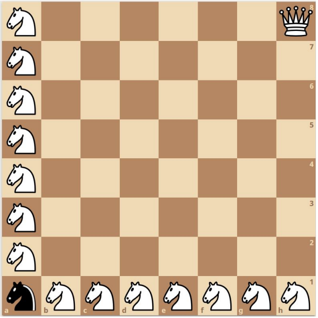
Motivating problem

In this question the axioms (or rules) are:

1. Knight starts at bottom left.
2. Knight can move two-right/one-up.
3. Knight can move two-up/one-right

Our goal is to see what we can deduce from these axioms. What positions are possible?





Motivating problem

Proposition

If the Knight can reach position (a, b), then the Knight can reach position $(a + 3, b + 3)$.

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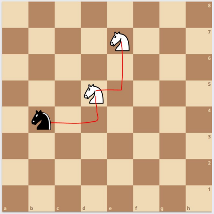
Proof.

From position (a, b) the Knight can use move 2 to go to position $(a + 2, b + 1)$. Then it can use move 3 to go to position $(a + 2 + 1, b + 1 + 2) = (a + 3, b + 3)$.



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Motivating problem

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Assume that the bottom left square is $(1, 1)$. The Knight cannot reach positions $(1, n)$ or $(n, 1)$ for any natural number $n > 1$.

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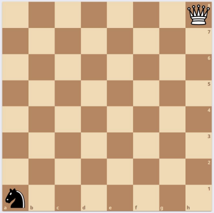
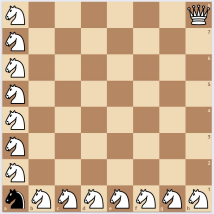
Proof.

The only way the knight can move is by simultaneously increasing both its x-coordinate <u>and</u> its y-coordinate. This means all positions $(1, n)$ or $(n, 1)$ (other than $(1, 1)$) are impossible to reach.





□

Axioms and theorems in math

In math, we will often start with axioms about particular math objects, and we will want to deduce what else is true about theses objects.
List of axioms tend to be:

- As short as possible.
- Not redundant.
- "Obviously" "true".

We will look at one set of axioms that are commonly used in math: the Peano axioms of arithmetic.

Peano Axioms

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If x, y, z are any integers, then

- A1 $x + y$ is an integer,
- A2 $x \cdot y$ is an integer,
- A3 $x + y = y + x$,
- A4 $x \cdot y = y \cdot x$,
- A5 $x \cdot (y + z) = x \cdot y + x \cdot z$,
- A6 $x + 0 = x$,
- A7 $x \cdot 1 = x$,
- A8 $x + (y + z) = (x + y) + z$,
- A9 $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
- A10 For every x there is an i such that $x + i = 0$. Denote this i as $-x$.

Peano Axioms

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$$\begin{aligned} 0 &= 0 + (-0) \qquad \text{By A10} \\ &= (-0) + 0 \qquad \text{By A3} \\ &= -0 \qquad \text{By A6} \end{aligned}$$

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Proposition

For every integer x, $0 \cdot x = 0$.

Notation

- The axioms are the starting assumptions in a rule set.
- A conjecture is something that you guess might be true.
- A proposition/fact/theorem are true statements that follow from the axioms. They have been proved.
- A lemma is minor proposition.
- A corollary is a proposition that follows from a previous known theorem. It is often a special case.

Reflection

- Go back and answer the Knight/King question.
- How would the Knight/King question change if we changed the axioms? (For example what if we allowed all "L-shaped" moves.)
- Do the Peano axioms capture everything true about the integers?