Introduction to Proofs - Mathematical Statements

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Identify a mathematical statement.
2. Change the truth value of a statement by changing the scope of variables.
3. Create a statement using $\forall, \exists$ quantifiers.
4. Translate a formal mathematical statement into English.
5. Distinguish mathematical statements with mixed quantifiers.

What is a mathematical statement?

Idea, not definition

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These are all mathematical statements:

- "$2 < 7$",
- "$-5=3$",
- "There is a real number z such that $\sin(z) = 10$."
- $\sqrt{2}$ is not an integer.

Non-examples

These are not mathematical statements.

- $\frac{12}{15}$,
- $\mathbb{N}$ or $\mathbb{Z}$,
- "$2 + 3$",
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Motivation

Our goal is to make conjectures and prove theorems. Only mathematical statements can be proved; non-mathematical statements don't assert anything.

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No! Because we don't know what x is, and the truth of this could change depending on the scope of x. For example, these are mathematical statements.

1. "$x^2 > 0$ when $x = 0$",
2. "$x^2 > 0$ when $x = 3$",
3. "$x^2 > 0$ for all real numbers x",
4. "$x^2 > 0$ for all natural numbers x",
5. "There is at least one real number x with $x^2 > 0$."

Quantifiers

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Words	Name	Symbol
For all, For any, Every	Universal Quantifier	$\forall$
There exists, There is, For some	Existential Quantifier	$\exists$

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If $P(x)$ is a property that x can have (such as "x is even" or "x is rational", we use this notation:

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Exercises

1. Finish the definition using quantifiers. An integer x is even if ...
2. Finish the definition using only quantifiers. A natural number n is composite if ...
3. Express this as a mathematical statement: For every positive real number, there is a number between it and 0.
4. **Harder**. Express this as a mathematical statement with mixed quantifiers: There is a smallest natural number.

Reflection

- Why do we care about mathematical statements? How are they related to conjectures and theorems?
- How would you feel if your grade in this course was "$\exists x$ such that $x/100$."?
- What does it take to show that the statement "$\forall x P(x)$" is false?
- Come up with a statement that uses three quantifiers.