Introduction to Proofs - Logical Equivalence

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Show that two formulas are logically equivalent.
2. Prove that a statement is a tautology or a contradiction.

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Example 1. P is logically equivalent to $\neg(\neg P)$.
Proof.

P	$\neg P$	$\neg(\neg P)$
T	F	
F	T	

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T	F	T
F	T	F

Column 1 (the truth values of P) and column 3 (the truth values of $\neg\neg P$) are identical, therefore P and $\neg\neg P$ are logically equivalent by definition.

Example 2

Example 2. $P \implies Q$ is logically equivalent to $(\neg Q) \implies (\neg P)$.
Proof.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$(\neg Q) \implies (\neg P)$
T	T				
T	F				
F	T				
F	F				

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Proof.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$(\neg Q) \implies (\neg P)$
T	T	T			
T	F	F			
F	T	T			
F	F	T			

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P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$(\neg Q) \implies (\neg P)$
T	T	T	F		
T	F	F	T		
F	T	T	F		
F	F	T	T		

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Proof.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$(\neg Q) \implies (\neg P)$
T	T	T	F	F	
T	F	F	T	F	
F	T	T	F	T	
F	F	T	T	T	

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Proof.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$(\neg Q) \implies (\neg P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Column 3 (the truth values of $P \implies Q$) and column 6 (the truth values of $(\neg Q) \implies (\neg P)$) are identical, therefore $P \implies Q$ is logically equivalent to $(\neg Q) \implies (\neg P)$.

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Proof.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$(\neg Q) \implies (\neg P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Column 3 (the truth values of $P \implies Q$) and column 6 (the truth values of $(\neg Q) \implies (\neg P)$) are identical, therefore $P \implies Q$ is logically equivalent to $(\neg Q) \implies (\neg P)$.

Definition (Contrapositive)

$\neg Q \implies \neg P$ is called the contrapositive of $P \implies Q$.

(Non-)Example 3

(Non-)Example 3

Show that $P \implies Q$ is <u>not</u> logically equivalent to $Q \implies P$.

Proof.

P	Q	$P \implies Q$	$Q \implies P$
T	T		
T	F		
F	T		
F	F		

□

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Show that $P \implies Q$ is not logically equivalent to $Q \implies P$.

Proof.

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	
T	F	F	
F	T	T	
F	F	T	

□

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Show that $P \implies Q$ is not logically equivalent to $Q \implies P$.

Proof.

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

□

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Show that $P \implies Q$ is not logically equivalent to $Q \implies P$.

Proof.

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

If P is true, and Q is false, then $P \implies Q$ is false, but $Q \implies P$ is true. □

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T	T	T	T
T	F	F	T
F	T	T	F
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If P is true, and Q is false, then $P \implies Q$ is false, but $Q \implies P$ is true. □

Human example. "If it rains, then it will be wet" is not the same as "If it is wet, then it rains.".

Tautologies and Contradictions

Exercise

Compute the truth tables of the statements $P \wedge (\neg P)$, and $P \vee (\neg P)$.

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T	F		
F	T		

Tautologies and Contradictions

Exercise

Compute the truth tables of the statements $P \wedge (\neg P)$, and $P \vee (\neg P)$.

P	$\neg P$	$P \wedge \neg P$	$P \vee (\neg P)$
T	F	F	
F	T	F	

Tautologies and Contradictions

Exercise

Compute the truth tables of the statements $P \wedge (\neg P)$, and $P \vee (\neg P)$.

P	$\neg P$	$P \wedge \neg P$	$P \vee (\neg P)$
T	F	F	T
F	T	F	T

Tautologies and Contradictions

Exercise

Compute the truth tables of the statements $P \wedge (\neg P)$, and $P \vee (\neg P)$.

P	$\neg P$	$P \wedge \neg P$	$P \vee (\neg P)$
T	F	F	T
F	T	F	T

Since $P \wedge \neg P$ is always false, it is called a <u>contradiction</u>. Since $P \vee \neg P$ is always true, it is called a <u>tautology</u>.

Exercises

Are the following statements contradictions, tautologies, or neither?

1. $P \implies P$
2. $P \implies \neg P$
3. $((\neg Q) \vee Q) \implies ((\neg Q) \wedge Q)$

Reflection

- Why might someone prefer to use the contrapositive of an implication, instead of the original implication?
- What happens when you take the contrapositive of the contrapositive?
- Construct your own tautology and contradiction.
- Do we need brackets when referring to $(P \implies Q) \implies R$, or is this the same as $P \implies (Q \implies R)$?
- In English, why would someone say " I'm not *not* eating cookies right now", instead of "I'm eating cookies right now".