Introduction to Proofs - Negation

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Create the negation of an English statement.
2. Formally negate a mathematical statement involving AND, NOT, IF/THEN.
3. Formally negate a mathematical statement involving multiple quantifiers

Motivation and first examples

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Note. When trying to get intuition for a negation of a statement R, it is helpful to ask "If someone said R to me, what would I need to know to be sure that they are lying?"

Negations of and/or

Example. Let P: "I am tall", and Q: "I play basketball".

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DeMorgan's laws

1. $\neg(P \wedge Q)$ is logically equivalent to $(\neg P) \vee (\neg Q)$.
2. $\neg(P \vee Q)$ is logically equivalent to $(\neg P) \wedge (\neg Q)$.

Exercise

Exercises

Express the following statements using and/or.

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2. $\neg(0 \leq x < 1)$.

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Solution 1. $(0 \leq x) \wedge (x < 1)$.
Solution 2.

$$\begin{aligned} \neg(0 \leq x < 1) &\Leftrightarrow \neg(0 \leq x \wedge x < 1) \\ &\Leftrightarrow \neg(0 \leq x) \vee \neg(x < 1) \qquad \text{by Demorgan's law} \\ &\Leftrightarrow x < 0 \vee 1 \leq x \end{aligned}$$

Negations of implications

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Example. Let P: "I get an A in my intro to proofs course", and Q: "I pass my intro to proofs course".
If someone says $P \implies Q$, the negation is $P \wedge (\neg Q)$: I got an A, but I didn't pass..

Negation of an implication

$\neg(P \implies Q)$ is logically equivalent to $P \wedge (\neg Q)$.

Negations of universal quantifiers

Example 1. "Every person in this course was born in Toronto."
Negation:

Negations of universal quantifiers

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Negation of universal quantifiers

$\neg(\forall x \in A, P(x))$ is logically equivalent to $(\exists x \in A)[\neg P(x)]$.

Proof technique for universal quantifiers

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To prove "$(\forall x \in A)[P(x)]$" is true, you must show that every x in A has the property $P(x)$.

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To prove "$(\forall x \in A)[P(x)]$" is true, you must show that every x in A has the property $P(x)$.

Note. No, one example is not enough to prove a universal statement.

Proof technique (negation of universal quantifiers)

To prove "$\neg(\forall x \in A)[P(x)]$" is true, you need to find only one example of an x in A that does not have the property $P(x)$. (This x is called a counterexample.)

Negations of existential quantifiers

Example 2. "There is a person in this course who is over 150 years old."
Negation:

Negations of existential quantifiers

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Negation: "Every person in this course is under (or exactly) 150 years old."

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Negation of existential quantifiers

$\neg(\exists x \in A, P(x))$ is logically equivalent to $(\forall x \in A)[\neg P(x)]$.

Proof technique for existential quantifiers

Proof technique (existential quantifiers)

To prove "$(\exists x \in A)[P(x)]$" is true, you must show that there is at least one x in A that has the property $P(x)$.

Note. Yes, one example is enough to prove an existential statement.

Proof technique for existential quantifiers

Proof technique (existential quantifiers)

To prove "$(\exists x \in A)[P(x)]$" is true, you must show that there is at least one x in A that has the property $P(x)$.

Note. Yes, one example is enough to prove an existential statement.

Proof technique (negation of existential quantifiers)

To prove "$\neg(\exists x \in A)[P(x)]$" is true, you need to show that all x in A that do not have the property $P(x)$.

Examples

Example

Negate these statements and then decide which is true: the original statement or the negation.

1. $(\forall x \in \mathbb{R})[x^2 > 0 \implies x > 0]$
2. $(\exists n \in \mathbb{N})[2^n > n^2]$

Example 1

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Negate these statements and then decide which is true: the original statement or the negation.

1. $(\forall x \in \mathbb{R})[x^2 > 0 \implies x > 0]$
2.

Solution 1.

$$\neg(\forall x \in \mathbb{R})[x^2 > 0 \implies x > 0] \equiv$$

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$$\neg(\forall x \in \mathbb{R})[x^2 > 0 \implies x > 0] \equiv (\exists x \in \mathbb{R})\neg[x^2 > 0 \implies x > 0]$$

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Solution 1.

$$\begin{aligned} \neg(\forall x \in \mathbb{R})[x^2 > 0 \implies x > 0] &\equiv (\exists x \in \mathbb{R})\neg[x^2 > 0 \implies x > 0] \\ &\equiv (\exists x \in \mathbb{R})[x^2 > 0 \wedge \neg(x > 0)] \\ &\equiv \end{aligned}$$

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This is true. Proof: Take $x = -5 < 0$, so $x^2 = 25 > 0$.

Example 2

Example

Negate these statements and then decide which is true: the original statement or the negation.

1.
2. $(\exists n \in \mathbb{N})[2^n > n^2]$

Solution 2.

$$\neg(\exists n \in \mathbb{N})[2^n > n^2] \equiv$$

$$\equiv$$

Example 2

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Negate these statements and then decide which is true: the original statement or the negation.

1.
2. $(\exists n \in \mathbb{N})[2^n > n^2]$

Solution 2.

$$\neg(\exists n \in \mathbb{N})[2^n > n^2] \equiv (\forall n \in \mathbb{N})\neg[2^n > n^2]$$
$$\equiv$$

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Negate these statements and then decide which is true: the original statement or the negation.

1.
2. $(\exists n \in \mathbb{N})[2^n > n^2]$

Solution 2.

$$\begin{aligned}\neg(\exists n \in \mathbb{N})[2^n > n^2] &\equiv (\forall n \in \mathbb{N})\neg[2^n > n^2] \\ &\equiv (\forall n \in \mathbb{N})[2^n \leq n^2]\end{aligned}$$

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Negate these statements and then decide which is true: the original statement or the negation.

1.
2. $(\exists n \in \mathbb{N})[2^n > n^2]$

Solution 2.

$$\begin{aligned} \neg(\exists n \in \mathbb{N})[2^n > n^2] &\equiv (\forall n \in \mathbb{N})\neg[2^n > n^2] \\ &\equiv (\forall n \in \mathbb{N})[2^n \leq n^2] \end{aligned}$$

This is false. The original statement is true. Proof: $n = 1 \in \mathbb{N}$ and $2^n = 2 > 1 = 1^2$.

Example 3 - Multiple Quantifiers

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Negate "$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[y < x]$".

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Solution.

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Negate "$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[y < x]$".

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$$\neg(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[y < x](\forall x \in \mathbb{R})\neg(\forall y \in \mathbb{R})[y < x]$$

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Negate "$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[y < x]$".

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$$\neg(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[y < x](\forall x \in \mathbb{R})\neg(\forall y \in \mathbb{R})[y < x]$$
$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\neg[y < x]$$

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Solution.

$$
\begin{aligned}
\neg(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[y < x] & (\forall x \in \mathbb{R})\neg(\forall y \in \mathbb{R})[y < x] \\
& (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\neg[y < x] \\
& (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[y \geq x]
\end{aligned}
$$

Example 3 - Multiple Quantifiers

Example 3

Negate "$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[y < x]$".

Solution.

$$\begin{aligned}
\neg(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[y < x] & (\forall x \in \mathbb{R})\neg(\forall y \in \mathbb{R})[y < x] \\
& (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\neg[y < x] \\
& (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[y \geq x]
\end{aligned}$$

This is true. Proof. Let $x \in \mathbb{R}$. Take $y = x + 1$. Note that $y = x + 1 \geq x$.

End boss

End boss exercise. Negate the following statement:

$$(\forall \epsilon > 0)(\exists \delta > 0)[0 < |x - 2| < \delta \implies |x^2 - 4| < \epsilon]$$

(Hint: You have all the tools you need to conquer this boss. Go slowly!)

Reflection

- Do you prefer to negate statements formally (using the process described here), or informally by "just thinking about it"?
- Negate $P_1 \wedge P_2 \wedge P_3$.
- How is deMorgan's law related to universal and existential statements?