Introduction to Proofs - Proof Strategies: Direct and Contrapositive

Prof Mike Pawliuk

UTM

May 21, 2020

Slides available at: mikepawliuk.ca

Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Prove an implication using a direct proof and a contrapositive.
2. State the converse and contrapositive of an implication.
3. Prove an equivalence.

Motivation

Motivation

We are now able to make mathematical statements (involving $\forall, \exists, \wedge, \vee, \implies, \neg, \Leftrightarrow$). Today we will see what it means to prove these statements.

Direct Proof

Proof Technique ($P \implies Q$, directly)

To prove a statement of the form $P \implies Q$. You assume P (is true). Deduce Q (by a series of logical, justified steps).

Direct Proof

Proof Technique ($P \implies Q$, directly)

To prove a statement of the form $P \implies Q$. You assume P (is true). Deduce Q (by a series of logical, justified steps).

Theorem

Let $x \in \mathbb{Z}$. If x is even, then x^2 is even.

Direct Proof

Proof Technique ($P \implies Q$, directly)

To prove a statement of the form $P \implies Q$. You assume P (is true). Deduce Q (by a series of logical, justified steps).

Theorem

Let $x \in \mathbb{Z}$. If x is even, then x^2 is even.

Proof.

Assume x is even.

□

Direct Proof

Proof Technique ($P \implies Q$, directly)

To prove a statement of the form $P \implies Q$. You assume P (is true). Deduce Q (by a series of logical, justified steps).

Theorem

Let $x \in \mathbb{Z}$. If x is even, then x^2 is even.

Proof.

Assume x is even. $\boxed{\longleftarrow \text{ ``Assume } P\text{.''}}$

$\square$

Direct Proof

Proof Technique ($P \implies Q$, directly)

To prove a statement of the form $P \implies Q$. You assume P (is true). Deduce Q (by a series of logical, justified steps).

Theorem

Let $x \in \mathbb{Z}$. If x is even, then x^2 is even.

Proof.

Assume x is even. $\boxed{\longleftarrow \text{ "Assume } P\text{."}}$
By definition, there is an integer k such that $x = 2k$. Notice $x^2 = (2k)^2 = 2(2k^2)$. Since k is an integer, so is $2k^2$.

□

Direct Proof

Proof Technique ($P \implies Q$, directly)

To prove a statement of the form $P \implies Q$. You assume P (is true). Deduce Q (by a series of logical, justified steps).

Theorem

Let $x \in \mathbb{Z}$. If x is even, then x^2 is even.

Proof.

Assume x is even. $\longleftarrow$ "Assume P."

By definition, there is an integer k such that $x = 2k$. Notice $x^2 = (2k)^2 = 2(2k^2)$. Since k is an integer, so is $2k^2$. Therefore, by definition, x^2 is even.

□

Direct Proof

Proof Technique ($P \implies Q$, directly)

To prove a statement of the form $P \implies Q$. You assume P (is true). Deduce Q (by a series of logical, justified steps).

Theorem

Let $x \in \mathbb{Z}$. If x is even, then x^2 is even.

Proof.

Assume x is even. $\boxed{\longleftarrow \text{ "Assume } P\text{."}}$
By definition, there is an integer k such that $x = 2k$. Notice $x^2 = (2k)^2 = 2(2k^2)$. Since k is an integer, so is $2k^2$.
Therefore, by definition, x^2 is even. $\boxed{\longleftarrow \text{ "Deduce } Q\text{."}}$

$\square$

Contrapositive

Proof Technique ($P \implies Q$, contrapositive)

To prove a statement of the form $P \implies Q$, we can instead prove the logically equivalent $(\neg Q) \implies (\neg P)$. Assume $\neg Q$. Deduce $\neg P$.

Contrapositive

Proof Technique ($P \implies Q$, contrapositive)

To prove a statement of the form $P \implies Q$, we can instead prove the logically equivalent $(\neg Q) \implies (\neg P)$. Assume $\neg Q$. Deduce $\neg P$.

Theorem

Let $x \in \mathbb{Z}$. If x^2 is even, then x is even.

Contrapositive

Proof Technique ($P \implies Q$, contrapositive)

To prove a statement of the form $P \implies Q$, we can instead prove the logically equivalent $(\neg Q) \implies (\neg P)$. Assume $\neg Q$. Deduce $\neg P$.

Theorem

Let $x \in \mathbb{Z}$. If x^2 is even, then x is even.

Proof.

We prove by contrapositive. Assume x is odd.

□

Contrapositive

Proof Technique ($P \implies Q$, contrapositive)

To prove a statement of the form $P \implies Q$, we can instead prove the logically equivalent $(\neg Q) \implies (\neg P)$. Assume $\neg Q$. Deduce $\neg P$.

Theorem

Let $x \in \mathbb{Z}$. If x^2 is even, then x is even.

Proof.

We prove by contrapositive. Assume x is odd. $\boxed{\longleftarrow \text{ "Assume } \neg Q\text{."}}$

□

Contrapositive

Proof Technique ($P \implies Q$, contrapositive)

To prove a statement of the form $P \implies Q$, we can instead prove the logically equivalent $(\neg Q) \implies (\neg P)$. Assume $\neg Q$. Deduce $\neg P$.

Theorem

Let $x \in \mathbb{Z}$. If x^2 is even, then x is even.

Proof.

We prove by contrapositive. Assume x is odd. $\boxed{\longleftarrow \text{ “Assume } \neg Q\text{.”}}$
By definition, there is an integer k such that $x = 2k + 1$. Notice $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since k is an integer, so is $2k^2 + 2k$.

□

Contrapositive

Proof Technique ($P \implies Q$, contrapositive)

To prove a statement of the form $P \implies Q$, we can instead prove the logically equivalent $(\neg Q) \implies (\neg P)$. Assume $\neg Q$. Deduce $\neg P$.

Theorem

Let $x \in \mathbb{Z}$. If x^2 is even, then x is even.

Proof.

We prove by contrapositive. Assume x is odd. $\boxed{\longleftarrow \text{"Assume } \neg Q\text{."}}$
By definition, there is an integer k such that $x = 2k + 1$. Notice $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since k is an integer, so is $2k^2 + 2k$.
Therefore, by definition, x^2 is odd.

□

Contrapositive

Proof Technique ($P \implies Q$, contrapositive)

To prove a statement of the form $P \implies Q$, we can instead prove the logically equivalent $(\neg Q) \implies (\neg P)$. Assume $\neg Q$. Deduce $\neg P$.

Theorem

Let $x \in \mathbb{Z}$. If x^2 is even, then x is even.

Proof.

We prove by contrapositive. Assume x is odd. $\boxed{\longleftarrow \text{ "Assume } \neg Q\text{."}}$
By definition, there is an integer k such that $x = 2k + 1$. Notice $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since k is an integer, so is $2k^2 + 2k$.
Therefore, by definition, x^2 is odd. $\boxed{\longleftarrow \text{ "Deduce } \neg P\text{."}}$

□

Different techniques

Question

How do I know which technique to use when proving $P \implies Q$?

Great question!

Different techniques

Question

How do I know which technique to use when proving $P \implies Q$?

Great question!

Exercise

Try to prove the previous implication (x^2 even implies x even) directly, and not by contrapositive. It's much harder.

Different techniques

Question

How do I know which technique to use when proving $P \implies Q$?

Great question!

Exercise

Try to prove the previous implication (x^2 even implies x even) directly, and not by contrapositive. It's much harder.

We'll get to a more detailed answer to this question after we see "proof by contradiction".

Converse

Recall

The converse of an implication is $Q \implies P$. It is NOT logically equivalent to $P \implies Q$.

Converse

Recall

The converse of an implication is $Q \implies P$. It is NOT logically equivalent to $P \implies Q$.

Example

"If I get an A, then I will pass" is not the same as "If I pass, then I will get an A."

Converse

Recall

The converse of an implication is $Q \implies P$. It is NOT logically equivalent to $P \implies Q$.

Example

"If I get an A, then I will pass" is not the same as "If I pass, then I will get an A."

- **Original statement**. $P \implies Q$.
- **Contrapositive**. $\neg Q \implies \neg P$. (Equivalent to original.)
- **Converse**. $Q \implies P$. (Not equivalent to original.)

Equivlences

Recall. $P \Leftrightarrow Q$ is logically equivlaent to "$(P \implies Q) \wedge (Q \implies P)$."

Equivlences

Recall. $P \Leftrightarrow Q$ is logically equivlaent to "$(P \implies Q) \wedge (Q \implies P)$."

Proof Technique ($P \Leftrightarrow Q$)

To prove $P \Leftrightarrow Q$ you do two things:

1. prove $P \implies Q$, and
2. prove $Q \implies P$.

Equivlences

Recall. $P \Leftrightarrow Q$ is logically equivlaent to "$(P \implies Q) \wedge (Q \implies P)$."

Proof Technique ($P \Leftrightarrow Q$)

To prove $P \Leftrightarrow Q$ you do two things:

1. prove $P \implies Q$, and
2. prove $Q \implies P$.

Proof Technique ($P \Leftrightarrow Q$) - Alternate

To prove $P \Leftrightarrow Q$ you can instead

1. prove $P \implies Q$, and
2. prove $\neg P \implies \neg Q$.

Equivalence proofs

Theorem

Let $x \in \mathbb{Z}$. x is even if and only if x^2 is even.

We proved this by showing

Equivalence proofs

Theorem

Let $x \in \mathbb{Z}$. x is even if and only if x^2 is even.

We proved this by showing $P \implies Q$ and $\neg P \implies \neg Q$.

Equivalence proofs

Theorem

Let $x \in \mathbb{Z}$. x is even if and only if x^2 is even.

We proved this by showing $P \implies Q$ and $\neg P \implies \neg Q$.

$P \Leftrightarrow Q \Leftrightarrow R$

You will often need to prove that three statements are equivalent. There are many ways to do this.

Strategy 1. Prove:

1. $P \implies Q$.
2. $Q \implies P$.
3. $Q \implies R$.
4. $R \implies Q$.

Equivalence proofs

Theorem

Let $x \in \mathbb{Z}$. x is even if and only if x^2 is even.

We proved this by showing $P \implies Q$ and $\neg P \implies \neg Q$.

$P \Leftrightarrow Q \Leftrightarrow R$

You will often need to prove that three statements are equivalent. There are many ways to do this.

Strategy 1. Prove:

1. $P \implies Q$.
2. $Q \implies P$.
3. $Q \implies R$.
4. $R \implies Q$.

Strategy 2. Prove:

1. $P \implies Q$.
2. $Q \implies R$.
3. $R \implies P$.

Exercise

Prove the following theorem that there are many equivalent ways to represent a rational number.

Theorem

Let x be a real number. The following are equivalent.

1. $(\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})[q \neq 0 \wedge x = \frac{p}{q}]$.
2. $(\exists a \in \mathbb{Z})(\exists b \in \mathbb{N})[x = \frac{a}{b}]$.
3. $(\exists c \in \mathbb{Z})(\exists d \in \mathbb{N})[x = \frac{c}{d} \wedge c, d$ have no common prime factors$]$.

Reflection

- What is the difference between a contrapositive and a converse?
- Create an example of an implication and construct its contrapositive and converse.
- If you have taken a linear algebra course you have seen a proof that 10 (or so) statements are equivalent to "a matrix is invertible". What proof strategy did they use?