TUTORIALS FOR MAT102 - FALL 2019

This document was originally created by Micheal Pawliuk in Fall 2019 while he was a professor at the University of Toronto Mississauga (UTM).

The materials were originally created for MAT102 Introduction to Proofs, a course a UTM.

Tutorial for Week 2 - TA Version

By the end of this tutorial, students should be able to:

(1) Make a conjecture relating the number of roots of a parabola to the number of roots of a transformation of that parabola.

(2) Decide if such a conjecture is true or false.

(3) Prove a true statement about parabolas, roots and transformations.

(4) Create a counterexample to a false statement about parabolas, roots and transformations.

Overview Thinking mathematically involves playing with mathematical objects, creating conjectures, and then trying to prove that they are true or false. In this tutorial, students will create conjectures about the roots of parabolas and how they interact with transformations.

Suggested Activities

Activity 1. In the first 5 minutes, introduce yourself, and explain how the tutorials will be run (What are **your** expectations? What are **their** expectations?).

Activity 2. Ask the students to write down the following:

- A general form of a parabola. (Both: $ax^2 + bx + c$ and $a(x - v)^2 + k$ are fine.)
- The discriminant of a parabola. ($b^2 - 4ac$)
- The number r is a root of a parabola $p(x)$ if ... ($p(r) = 0$)
- The statement of the Quadratic formula.

Activity 3. Describe algebraically and geometrically how the following transformations work:

- Horizontal shifts. ($p(x - 2)$)
- Vertical shifts. ($p(x) + 2$)
- Horizontal stretches. ($p(2x)$)
- Vertical stretches. ($2p(x)$)
- Reflection across x-axis. ($-p(x)$)
- Reflection across y-axis. ($p(-x)$)

Activity 4. Give students 15 minutes to come up with as many conjectures about how these transformations interact with the number of roots of the parabola. Let them write their conjectures on the board. Start it off by writing the following conjectures:

(1) A horizontal shift doesn't change the number of roots of a parabola. (True.)

(2) A vertical shift doesn't change the number of roots of a parabola. (False.)

Activity 5. Pick two of the conjectures and explain why you think they are true or false (possibly through your **geometric visualization**). Then provide a proof for the true one (through **algebra**), and a counterexample for the false one.

For example, to show "A horizontal shift doesn't change the number of roots of a parabola." start with a general parabola $p(x) = ax^2 + bx + c$, and show that $p(x - t) = a(x - t)^2 + b(x - t) + c$ has the same discriminant as $p(x)$ and therefore the same number of roots.

Activity 6. Leave the rest of the time for students to prove, or provide counterexamples, to the rest of the conjectures.

Tutorial for Week 3 - TA Version

By the end of this tutorial, students should be able to:

(1) Prove a basic fact about rational numbers using proof by "definition unwinding".

(2) Determine if a conjecture about rational numbers is true or false.

(3) Determine if an expression is a mathematical statement.

Overview This week students have Quiz 1 on Thursday (which covers Block 1) and an assignment due on Friday (which covers Blocks 1 and 2). This tutorial will be relevant for those assessments.

Suggested Activities

Activity 0. Check-in with the students: How is the term going so far? How do they feel about the upcoming quiz? How is Problem Set 1 going? (Remind students that they can ask/answer questions on Piazza.)

Activity 1. Prove (by definition unwinding) that "If $-\sqrt{2}$ is rational, then $\sqrt{2}$ is rational". Is this actually asserting that $\sqrt{2}$ is rational (which is false)?

Discuss how this can be **generalized** to "If x is rational, then $-x$ is rational." Students should always be on the lookout for generalizations.

Activity 2. Which of the following conjectures are true? If they are true, provide a proof (by definition unwinding). If they are false, provide a counterexample.

(1) The product of any two rational numbers is rational.

(2) The product of any two irrational numbers is irrational.

(3) If x is rational and $x+y$ is rational, then y is rational.

Ask students to reflect on their strategies for determining if a conjecture is true or false. There are no methods that are guaranteed to succeed, but there are methods that are guaranteed to fail.

Activity 3. Which of the following are mathematical statements?

(1) x^2+6x+9

(2) $(x+3)^2=0$

(3) $x^2+6x+9=(x+3)^2$

(4) $(-3)^2+6(-3)+9$

(5) -3 is a root of x^2+6x+9

(6) -3 is a root of $x^2+6x+9=0$

(7) $(x+3)^2$ has two roots.

MAT102H5 F 2019 - Tutorial Handout - Week 3

By the end of this tutorial, students should be able to:

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(2) Determine if a conjecture about rational numbers is true or false.

(3) Determine if an expression is a mathematical statement.

Activity 1. Prove (by definition unwinding) that "If $-\sqrt{2}$ is rational, then $\sqrt{2}$ is rational". Is this actually asserting that $\sqrt{2}$ is rational?

Activity 2. Which of the following conjectures are true? If they are true, provide a proof (by definition unwinding). If they are false, provide a counterexample.

(1) The product of any two rational numbers is rational.

(2) The product of any two irrational numbers is irrational.

(3) If x is rational and $x + y$ is rational, then y is rational.

Activity 3. Which of the following are mathematical statements?

(1) $x^2 + 6x + 9$

(2) $(x + 3)^2 = 0$

(3) $x^2 + 6x + 9 = (x + 3)^2$

(4) $(-3)^2 + 6(-3) + 9$

(5) $(-3)^2 + 6(-3) + 9 = 0$

(6) -3 is a root of $x^2 + 6x + 9$

(7) -3 is a root of $x^2 + 6x + 9 = 0$

(8) $(x + 3)^2$ has two roots.

Tutorial for Week 4 - TA Version

By the end of this tutorial, students should be able to:

(1) Express the converse and contrapositive of a statement.

(2) Identify logically equivalent statements.

(3) Negate complicated mathematical statements (such as the limit definition from calculus).

Overview Students just had Quiz 1 and Problem Set 1 last week. This week we're full into informal logic (statements, negations, logical equivalences) but we haven't got to basic proof techniques yet (direct proof, contrapositive, contradiction). The punchline today is to negate the limit definition from calculus (although we do not expect them to have seen calculus already).

Suggested Activities

Activity 0. Check-in with the students: How is the term going so far? How did Quiz 1 go? How did Problem Set 1 go? What are they going to do differently/the same for future quizzes/problem sets?

Activity 1. Let R be the statement "If I live in Toronto, then I live in Canada."

(1) Write R in the form $P \Rightarrow Q$.

(2) Write the converse of R. Is it a true statement?

(3) Write the contrapositive of R. Is it a true statement?

(4) Give an example of an "If/then" statement that is logically equivalent to its converse.

Activity 2. Which of the following are logically equivalent for mathematical statements P, Q, R?

(1) "$P \vee P$" and "P".

(2) "$P \wedge P$" and "P".

(3) "$P \vee P \wedge P$" and "$P \wedge P \vee P$".

(4) "$(P \Rightarrow Q) \Rightarrow R$" and "$P \Rightarrow (Q \Rightarrow R)$"

Activity 3. Negate the following sentences. Note that $\mathbb{R}^+$ is the set of all positive real numbers.

(1) "$(\forall \epsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)[P]$"

(2) "$P \Rightarrow Q$"

(3) "$|f(x)| < \epsilon$"

(4)
$$\text{“}(\forall \epsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)[(0 < |x-2| < \delta) \Rightarrow (|x^2-4| < \epsilon)]\text{”}$$

MAT102H5 F 2019 - Tutorial Handout - Week 4

By the end of this tutorial, students should be able to:

(1) Express the converse and contrapositive of a statement.

(2) Identify logically equivalent statements.

(3) Negate complicated mathematical statements.

Suggested Activities

Activity 1. Let R be the statement "If I live in Toronto, then I live in Canada."

(1) Write R in the form $P \Rightarrow Q$.

(2) Write the converse of R. Is it a true statement?

(3) Write the contrapositive of R. Is it a true statement?

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(2) "$P \wedge P$" and "P".

(3) "$P \vee P \wedge P$" and "$P \wedge P \vee P$".

(4) "$(P \Rightarrow Q) \Rightarrow R$" and "$P \Rightarrow (Q \Rightarrow R)$"

Activity 3. Negate the following sentences. Note that $\mathbb{R}^+$ is the set of all positive real numbers.

(1) "$(\forall \epsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)[P]$"

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(4)

$$\text{“}(\forall \epsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)[(0 < |x-2| < \delta) \Rightarrow (|x^2-4| < \epsilon)]\text{”}$$

Tutorial for Week 5 - TA Version

By the end of this tutorial, students should be able to:

(1) Prove an implication using a direct proof, contrapositive and contradiction.

(2) Decide which proof technique (Direct, contrapositive, contradiction) is most appropriate.

Overview Students have just learned the basic proof techniques (direct, contrapositive and contradiction but not induction). In this tutorial they will practice identifying when these techniques are applicable, and how to use them.

Everything this tutorial is centred around divisibility and even/odd numbers. This is to show off how mathematicians "develop a theory" by proving as much as they can in a given subject area. Students will get an opportunity to do this later in the course with subjects like Equivalence Relations and Countability.

Suggested Activities

Activity 0. Check in with the students. How is the course going? Have a short discussion about Problem Set 1: How did they find it? What are your observations as a TA.

Activity 1. State the formal definitions of an even number x, "$(\exists m \in \mathbb{Z})[x = 2m]$", and an odd number y, "$(\exists m \in \mathbb{Z})[x = 2m + 1]$".

Ask the students to prove the following statements. (Aim for air-tight proofs by definition unwinding.)

(1) 2019 is odd. (Proof: $2019 = 2(1009) + 1$.)

(2) $(\forall x \in \mathbb{Z})[x \text{ is even } \Leftrightarrow x - 3 \text{ is odd}]$,

Proof. Let $x \in \mathbb{Z}$. We prove $\Rightarrow$ first, then $\Leftarrow$.

Suppose x is even. So there is an integer m such that $x = 2m$. Note that

$$x - 3 = 2m - 3 = 2m - 4 + 1 = 2(m - 2) + 1$$

Since $m - 2 \in \mathbb{Z}$, we have that $x - 3$ is odd by definition.

Now the other direction. Suppose $x - 3$ is odd. So there is an integer m such that $x - 3 = 2m + 1$. Note that

$$x = x - 3 + 3 = (2m + 1) + 3 = 2(m + 2)$$

Since $m + 2 \in \mathbb{Z}$, we have that x is even by definition. □

(3) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})[y \text{ is even and } x < y]$,

Proof. Let $x \in \mathbb{Z}$. Note that either $x + 1$ or $x + 2$ will be even, and that will work for y. (This proof will assume that every number is either even or odd, but we will prove that later in this tutorial.)

If x is even, then $x + 1$ is clearly odd ($x = 2m$, so $x + 1 = 2m + 1$). If x is odd, then $x + 1$ is even. ($x = 2m + 1$, so $x + 1 = 2m + 1 + 1 = 2(m + 1)$.) □

(4) Let $x \in \mathbb{Z}$. If the final digit of x is $0, 2, 4, 6$, or 8, then x is even. (Use the fact that a number can always be represented as $x = 10n + m$ where $m \in \{0, 1, 2, \ldots, 9\}$ and $n \in \mathbb{Z}$.)

Proof. Essentially, $10n = 2(5n)$ is even, and if the final digit is one of $0, 2, 4, 6, 8$, then you can represent x as an even. □

Activity 2. Discuss the statements "Every integer is even or odd" and "No integer is both even and odd".

(1) Are these the same statement? (No. The first one is like "everyone has a sandwich", and the second one is like "No one has two sandwiches.")

(2) Do they require proofs from the definitions we gave? (Yes. Just because a number has a certain form, does not automatically exclude it from having an additional second form.)

(3) How does this definition of "odd" differ from the one in the textbook? (The textbook definition of odd is "not even". This means that both previous statements are immediate consequences of these definitions.)

Prove, by contradiction, that $(\forall x \in \mathbb{Z})[x \text{ is even} \Rightarrow x \text{ is not odd}]$. Prove the other implication as well. Conclude that no integer is both even and odd.

Activity 3. Decide which technique (direct, contrapositive or contradiction) to use to prove each statement.

(1) Let $a, b, c \in \mathbb{N}$. If $a|b$ and $b|c$ then $a|c$. (Direct.)

(2) Let $x \in \mathbb{R}$. If x is irrational, then $\sqrt{x}$ is irrational. (Contrapositive. If they try contradiction, show them how it is really a contrapositive.)

(3) There is no $q \in \mathbb{Q}$ that is a solution to $x^2 + x + 1 = 0$. (Contradiction. Assume there is a solution, and it is in reduced form. Plug it in and show, using parity cases, that the LH of the equation is always odd. A contradiction.)

(4) There are no natural numbers $n \geq 3$ with $(n-1)|n$. (Contrapositive. This is hard!)

Proof. Suppose that $n \in \mathbb{N}$ is such that $(n-1)|n$. By definition, there is an $x \in \mathbb{Z}$ such that $n = x(n-1)$. Note that $x \neq 1$, since $n \neq n-1$.

By rearranging, we get $n = \dfrac{x}{x-1}$. It is easy to convince yourself that this quantity is ≤ 2. □

MAT102H5 F 2019 - Tutorial Handout - Week 5

By the end of this tutorial, students should be able to:

(1) Prove an implication using a direct proof, contrapositive and contradiction.

(2) Decide which proof technique (Direct, contrapositive, contradiction) is most appropriate.

Suggested Activities

Activity 1. Prove the following statements. (Aim for air-tight proofs by definition unwinding.)

(1) 2019 is odd.

(2) $(\forall x \in \mathbb{Z})[x \text{ is even } \Leftrightarrow x - 3 \text{ is odd}]$,

(3) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})[y \text{ is even and } x < y]$,

(4) Let $x \in \mathbb{Z}$. If the final digit of x is $0, 2, 4, 6$, or 8, then x is even. (Use the fact that a number can always be represented as $x = 10n + m$ where $m \in \{0, 1, 2, \dots, 9\}$ and $n \in \mathbb{Z}$.)

Activity 2. Discuss the statements "Every integer is even or odd" and "No integer is both even and odd".

(1) Are these the same statement?

(2) Do they require proofs from the definitions we gave?

(3) How does this definition of "odd" differ from the one in the textbook?

Prove, by contradiction, that $(\forall x \in \mathbb{Z})[x \text{ is even} \Rightarrow x \text{ is not odd}]$.

Prove the other implication as well.

Conclude that no integer is both even and odd.

Activity 3. Decide which technique (direct, contrapositive or contradiction) to use to prove each statement.

(1) Let $a, b, c \in \mathbb{N}$. If $a|b$ and $b|c$ then $a|c$.

(2) Let $x \in \mathbb{R}$. If x is irrational, then $\sqrt{x}$ is irrational.

(3) There is no $q \in \mathbb{Q}$ that is a solution to $x^2 + x + 1 = 0$.

(4) There are no natural numbers $n \geq 3$ with $(n-1)|n$.

Tutorial for Week 6 - TA Version

By the end of this tutorial, students should be able to:

(1) Compute the image of a function applied to various sets.

(2) Conjecture set identities from Venn diagrams.

(3) Prove a set identity by definition unwinding.

(4) Find counterexamples to false set identities.

Overview The students have their second Quiz on Thursday October 10 (this week); it covers Blocks 2 to Block 4 (sets and functions). Today's tutorial will contain material about sets and functions that will be relevant for the quiz. In particular, in this tutorial they will use three different perspectives to get intuition about potential set identities.

Feel free to cut some of the activities short if the students have specific questions they would like answered (relating to Quiz 2).

Suggested Activities

Activity 0. Check in with the students. How is the course going? Have a short discussion about Problem Set 1: How did they find it? What are your observations as a TA.

Activity 1. Consider the function $f : \{1, 2, 3\} \to \{1, 2, 3\}$, where $f(1) = 1, f(2) = 1$ and $f(3) = 2$.

(1) Draw the arrow diagram for f.

(2) What is the range of f?

(3) Find two different non-empty subsets of $A, B \subseteq \{1, 2, 3\}$ such that $f(A) = f(B)$.

(4) Find two different non-empty subsets of $C, D \subseteq \{1, 2, 3\}$ such that $f(C) \neq f(D)$.

Activity 2. Draw the following sets using Venn-diagrams:

(1) $A \cap B$

(2) $A \setminus (B \setminus A)$

(3) $A \setminus (A \setminus B)$

(4) $A \cup (B \cap C)$

(5) $A \cap (B \cup C)$

(6) $(A \cup B) \cap (A \cup C)$

Based on your drawings, conjecture two set identities. Prove one of them by double-subset and definition unwinding.

Theorem 1. *Let A, B, C be sets.* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow x \in A \vee (x \in B \cap C) \\ &\Leftrightarrow x \in A \vee (x \in B \wedge x \in C) \\ &\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \\ &\Leftrightarrow x \in A \cup B \wedge x \in A \cup C \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C) \end{aligned}$$

□

Note that the key step is that "$\vee$ distributes over $\wedge$".

This is the harder one:

Theorem 2. *Let A, B be sets. $A \setminus (A \setminus B) = A \cap B$.*

Proof.

$$\begin{aligned} x \in A \setminus (A \setminus B) &\Leftrightarrow x \in A \wedge (x \notin A \setminus B) \\ &\Leftrightarrow x \in A \wedge \neg(x \in A \setminus B) \\ &\Leftrightarrow x \in A \wedge \neg(x \in A \wedge x \notin B) \\ &\Leftrightarrow x \in A \wedge (\neg(x \in A) \vee \neg(x \notin B)) \\ &\Leftrightarrow x \in A \wedge (x \notin A \vee x \in B) \\ &\Leftrightarrow (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B) \\ &\Leftrightarrow \text{False} \vee (x \in A \wedge x \in B) \\ &\Leftrightarrow x \in A \wedge x \in B \\ &\Leftrightarrow x \in A \cap B \end{aligned}$$

□

Activity 3. Find sets A, B, C where the following identities are false.

(1) $A \cap B = A$

(2) $B \setminus A = A \setminus B$

(3) $(A \setminus B) \cup (B \setminus A) = A \cup B$

For the previous activity, encourage them to draw venn diagrams. If the pictures don't agree, that tells them where to look for a counterexample.

Activity 4. Let A, B be sets. Prove that $A \setminus B = \emptyset$ if and only if $A \subseteq B$.

Proof. $\Rightarrow$, by contrapositive. Suppose $A \not\subseteq B$. So there is an $x \in A$ with $x \notin B$. Thus $x \in A \setminus B$. So $A \setminus B \neq \emptyset$.

$\Leftarrow$, by contrapositive. Suppose $A \setminus B \neq \emptyset$. So let $x \in A \setminus B$. That is $x \in A$ and $x \notin B$. So this x also shows that $A \not\subseteq B$. □

MAT102H5 F 2019 - Tutorial Handout - Week 6

By the end of this tutorial, students should be able to:

(1) Compute the image of a function applied to various sets.

(2) Conjecture set identities from Venn diagrams.

(3) Prove a set identity by definition unwinding.

(4) Find counterexamples to false set identities.

Suggested Activities

Activity 1. Consider the function $f : \{1, 2, 3\} \to \{1, 2, 3\}$, where $f(1) = 1, f(2) = 1$ and $f(3) = 2$.

(1) Draw the arrow diagram for f.

(2) What is the range of f?

(3) Find two different non-empty subsets of $A, B \subseteq \{1, 2, 3\}$ such that $f(A) = f(B)$.

(4) Find two different non-empty subsets of $C, D \subseteq \{1, 2, 3\}$ such that $f(C) \neq f(D)$.

Activity 2. Draw the following sets using Venn-diagrams:

(1) $A \cap B$

(2) $A \setminus (B \setminus A)$

(3) $A \setminus (A \setminus B)$

(4) $A \cup (B \cap C)$

(5) $A \cap (B \cup C)$

(6) $(A \cup B) \cap (A \cup C)$

Based on your drawings, conjecture two set identities. Prove one of them by double-subset and definition unwinding.

Activity 3. Find sets A, B, C where the following identities are false.

(1) $A \cap B = A$

(2) $B \setminus A = A \setminus B$

(3) $(A \setminus B) \cup (B \setminus A) = A \cup B$

Activity 4. Let A, B be sets. Prove that $A \setminus B = \emptyset$ if and only if $A \subseteq B$.

Tutorial for Week 7 - TA Version

By the end of this tutorial, students should be able to:

(1) Identify the (set theoretic) type of a relation.

(2) Check if a given relation is transitive, symmetric, or reflexive.

(3) Partition a space using equivalence classes.

Overview The students have PS3 due on Friday October 25th. For this tutorial, students will explore the structures that emerge from equivalence relations. The first exercises should have them moving around and talking to each other. In this way they can see and internalize what an equivalence relation looks like.

In the second half they will look at the collection of all equivalence relations on a set. This has some beautiful and rich structure (and shows off some of the fun of doing mathematics). At some point in the tutorial students should realize that it's more convenient to describe equivalence relations by using partitions as opposed to explicitly listing out pairs. That is a major insight!

Suggested Activities

Activity 0. Check in with the students. How was reading week? How is PS3 going?

For the next activities there's some moving around. Also, don't forget to include yourself in the set X!

Activity 1. Let X be the collection of all people in the room. For $p, q \in X$, define $(p, q) \in R$ if and only if p and q share at least one letter in their name.

For example, (John, Mike) $\notin R$, but (Mike, Ali) $\in R$ because they both have an i in their name.

(1) Show that this relation is symmetric and reflexive.

(2) Show that this relation is not transitive by finding a counterexample of three people.

Activity 2. Let X be the collection of all people in the room. For $p, q \in X$, define $(p, q) \in S$ if and only if p and q share the first letter of their name.

For example, (Mike, Ali) $\notin S$, but (Ali, Amir) $\in S$.

(1) Show that this is an equivalence relation.

(2) Find your equivalence class: [you].

(3) Is anyone in more than one equivalence class?

Have a discussion afterwards about how each equivalence class could have a "team name" that makes sense to it. What's the most number of equivalence classes that could possibly form? (26). Namely, everyone in an equivalence class will have the same first letter, so that's a natural choice for the "team name" of each equivalence class.

Activity 3. Let $X = \{1, 2, 3\}$. Show that each of the following relations are not equivalence relations on X. What new pairs can be included so that each relation becomes an equivalence relation?

(1) $R_1 = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$

(2) $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

(3) $R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

(4) $R_4 = \{(1, 1), (2, 2)\}$

Have a discussion about what "shapes" (literally directed graphs) of equivalence relations are possible on $\{1, 2, 3\}$. There is the complete relation $X \times X$ (which has one part), the singleton/discrete relation $\{(1, 1), (2, 2), (3, 3)\}$ (which has three parts) and the three relations that contain 2 parts.

Activity 4. Let $X = \{1, 2, 3\}$. List all five possible equivalence relations $R_1, R_2, \ldots, R_5$ on X. Use a (lattice) diagram to indicate which relations are subsets of each other. For example $\{(1, 1), (2, 2), (3, 3)\} \subseteq \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$.

Challenge. Repeat this exercise for $X = \{1, 2, 3, 4\}$; it has 15 possible equivalence relations. What kinds of symmetry (and non-symmetry) does the lattice have?

Here are the partitions:

- 1 part: 1234
- 2 parts: 1/234, 2/134. 3/124. 4/124. 12/34. 13/24, 14/23
- 3 parts: 1/2/34. 1/3/24, 1/4/23, 2/3/14, 2/4/13, 3/4/12
- 4 parts: 1/2/3/4

There are two important lessons here which students will discover, but may not be able to articulate:

(1) **Good naming of things is important for organizing them**. The actual lattice of equivalence relations isn't too complicated. The hard part is representing the relations in a clear, consistent and manageable way. You may wish to come up with notation, as a tutorial, that works for this problem. For example, $R_{1/4/23}$ could correspond to the relation where $1, 4$ are singletons, and $2, 3$ are related.

(2) **Partitions and equivalence relations are related ideas**.

If students are interested, the number of partition "shapes" of $\{1, 2, \ldots, n\}$ is called the partition number of n. You can direct students to the Wikipedia article for "Partition (number theory)".

`https://en.wikipedia.org/wiki/Partition_(number_theory)`

MAT102H5 F 2019 - Tutorial Handout - Week 7

By the end of this tutorial, students should be able to:

(1) Identify the (set theoretic) type of a relation.

(2) Check if a given relation is transitive, symmetric, or reflexive.

(3) Partition a space using equivalence classes.

Suggested Activities

Activity 1. Let X be the collection of all people in the room. For $p, q \in X$, define $(p, q) \in R$ if and only if p and q share at least one letter in their name.

For example, (John, Mike) $\notin R$, but (Mike, Ali) $\in R$ because they both have an i in their name.

(1) Show that this relation is symmetric and reflexive.

(2) Show that this relation is not transitive by finding a counterexample of three people.

Activity 2. Let X be the collection of all people in the room. For $p, q \in X$, define $(p, q) \in S$ if and only if p and q share the first letter of their name.

For example, (Mike, Ali) $\notin S$, but (Ali, Amir) $\in S$.

(1) Show that this is an equivalence relation.

(2) Find your equivalence class: [you].

(3) Is anyone in more than one equivalence class?

Activity 3. Let $X = \{1, 2, 3\}$. Show that each of the following relations are not equivalence relations on X. What new pairs can be included so that each relation becomes an equivalence relation?

(1) $R_1 = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$

(2) $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

(3) $R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

(4) $R_4 = \{(1, 1), (2, 2)\}$

Activity 4. Let $X = \{1, 2, 3\}$. List all five possible equivalence relations $R_1, R_2, \ldots, R_5$ on X. Use a (lattice) diagram to indicate which relations are subsets of each other. For example $\{(1, 1), (2, 2), (3, 3)\} \subseteq \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$.

Activity 5. Challenge. Repeat the previous exercise for $X = \{1, 2, 3, 4\}$; it has 15 possible equivalence relations. What kinds of symmetry (and non-symmetry) does the lattice have?

Tutorial for Week 8 - TA Version

By the end of this tutorial, students should be able to:

(1) Use the triangle inequality to prove a bounding argument.

(2) Apply the AMGM inequality to prove new inequalities.

Overview The midterm is coming up on Thursday (October 31). Today's tutorial activities are meant to give practice with inequalities (there's a long-answer question about inequalities on the midterm). The major goal here is for them to get to the point where they can prove the n-term AMGM:

$$(\forall n \in \mathbb{N})(\forall x_1, \ldots, x_n \geq 0)\left[\frac{x_1 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}\right]$$

and the weighted version of the AMGM:

$$(\forall x, y \geq 0)(\forall a, b \geq 0)[a + b = 1 \Rightarrow ax + by \geq x^a y^b]$$

Suggested Activities

Activity 0. Check in with the students. How do they feel about the midterm? How did PS3 go?

Activity 1. Bounding arguments. Use the triangle inequality to find a number $M \in \mathbb{R}$ such that:

$$(\forall x \in [-2, 0])[|-x^3 - 3x^2 + 6x + 1| \leq M]$$

It is known (e.g. through calculus) that the smallest M that works is 15. Explain why your M is larger than 15.

Proof.

$$\begin{aligned}
|-x^3 - 3x^2 + 6x + 1| &\leq |-x^3| + |3x^2| + |-6x| + |1| \\
&\leq |x^3| + 3|x^2| + 6|x| + 1 \\
&\leq |x|^3 + 3|x|^2 + 6|x| + 1 \\
&\leq 2^3 + 3 \cdot 2^2 + 6 \cdot 2 + 1 = 33
\end{aligned}$$

Note that the largest possible value of $|x|, |x|^2$ and $|x|^3$ is when $x = -2$. The reason this value is less than 11 is because we are ignoring all possible cancellation of the terms by using the triangle inequality. The number 11 takes the cancellation into account. □

Activity 2. Use AMGM to prove:

$$(\forall x \geq 0)[x(x + 2) \leq (x + 1)^2]$$

Proof. Note that $x + 1$ is the average (arithmetic mean) of x and $x + 2$. Also, both terms are non-negative. Therefore:

$$\sqrt{x(x + 2)} \leq \frac{x + (x + 2)}{2} = x + 1$$

Squaring both sides gives:

$$x(x + 2) \leq (x + 1)^2$$

□

You can prompt them to think about whether this inequality is true for <u>all</u> $x \in \mathbb{R}$. It is! However, special attention is required when $-2 < x < 0$ because the AMGM no longer applies (but obviously the LHS will be negative while the RHS is positive).

Activity 3. Use AMGM (3 times!) to prove: $\forall x, y, z, w \geq 0$

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$$

Discuss what other versions of AMGM can be proved using this technique. (Notably this works for $n =$ a power of 2.)

Proof. Applying AMGM to x, y gives:

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Applying AMGM to z, w gives:

$$\frac{z+w}{2} \geq \sqrt{zw}.$$

Applying AMGM to $\frac{x+y}{2}$ and $\frac{z+w}{2}$ gives:

$$\frac{\frac{x+y}{2} + \frac{z+w}{2}}{2} \geq \sqrt{\frac{x+y}{2}\frac{z+y}{2}} \geq \sqrt{\sqrt{xy}\sqrt{xy}} = \sqrt[4]{xyzw}.$$

□

Activity 4. Use the previous result to prove: $\forall x, y \geq 0$

$$\frac{x}{4} + \frac{3y}{4} \geq \sqrt[4]{xy^3}$$

Discuss that this is a special version of the weighted AMGM. The LHS is an average, but it's not an equal average. We add weight to the y term.

Activity 5. Use AMGM for 4 terms, with $w = \sqrt[3]{xyz}$ to prove: $\forall x, y, z \geq 0$

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

This one is challenging and will require you to give hints.

Proof. Applying the 4 term AMGM gives:

$$\frac{x+y+z+\sqrt[3]{xyz}}{4} \geq \sqrt[4]{xyz\sqrt[3]{xyz}}$$

The RHS becomes:

$$\sqrt[4]{xyz\sqrt[3]{xyz}} = \sqrt[4]{(xyz)^{\frac{4}{3}}} = \sqrt[3]{xyz}$$

So then we have:

$$\frac{x+y+z+\sqrt[3]{xyz}}{4} \geq \sqrt[3]{xyz}$$

Multiplying both sides by $\frac{4}{3}$ gives us:

$$\frac{x+y+z+\sqrt[3]{xyz}}{3} \geq \frac{4}{3}\sqrt[3]{xyz}$$

And moving the $\frac{\sqrt[3]{xyz}}{3}$ to the RHS gives us our desired inequality. □

The following three activities are challenges for interested students. You don't need to discuss these with the whole tutorial.

Activity 6. Challenge. Generalize your previous argument to show that if you know AMGM for n terms is true (and $n > 2$), then you can prove AMGM for $n - 1$ terms.

Activity 7. Challenge. Combine everything you've done here to conclude that for any $n \in \mathbb{N}$ the AMGM for n terms is true.

Idea: Prove it for all powers of 2. Then use the fact that you can step down n from the previous activity.

Activity 8. Challenge. Let $n \in \mathbb{N}$. Suppose that $a_1, \ldots, a_n$ are positive rationals such that $a_1 + \ldots + a_n = 1$. Prove, using the previous result, that $\forall x_1, \ldots, x_n \geq 0$ that

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \geq x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

This is the weighted AMGM (with rational weights).

Idea: Find a common denominator for the weights, and then this becomes a special case of the previous activity. This is a generalization of activity 3.

MAT102H5 F 2019 - Tutorial Handout - Week 8

Suggested Activities

Activity 1. Bounding arguments. Use the triangle inequality to find a number $M \in \mathbb{R}$ such that:

$$(\forall x \in [-2,0])[|-x^3 - 3x^2 + 6x + 1| \leq M]$$

It is known (e.g. through calculus) that the smallest M that works is 15. Explain why your M is larger than 15.

Activity 2. Use AMGM to prove:

$$(\forall x \geq 0)[x(x+2) \leq (x+1)^2]$$

Activity 3. Use AMGM (3 times!) to prove: $\forall x, y, z, w \geq 0$

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$$

Hint:

$$\frac{\frac{x+y}{2} + \frac{z+w}{2}}{2} = \frac{x+y+z+w}{4}.$$

Activity 4. Use the previous result to prove: $\forall x, y \geq 0$

$$\frac{x}{4} + \frac{3y}{4} \geq \sqrt[4]{xy^3}$$

Activity 5. Use AMGM for 4 terms, with $w = \sqrt[3]{xyz}$ to prove: $\forall x, y, z \geq 0$

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

Activity 6. Challenge. Generalize your previous argument to show that if you know AMGM for n terms is true (and $n > 2$), then you can prove AMGM for $n-1$ terms.

Activity 7. Challenge. Combine everything you've done here to conclude that for any $n \in \mathbb{N}$ the AMGM for n terms is true.

Activity 8. Challenge. Let $n \in \mathbb{N}$. Suppose that $a_1, \ldots, a_n$ are positive rationals such that $a_1 + \ldots + a_n = 1$. Prove, using the previous result, that $\forall x_1, \ldots, x_n \geq 0$ that

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

This is the weighted AMGM (with rational weights).

Tutorial for Week 9 - TA Version

By the end of this tutorial, students should be able to:

(1) Generalize a statement about a particular number, to a statement $P(n)$.

(2) Prove that a given explicit formula generates a given recursively defined sequence.

(3) Improve the presentation of a proof by induction.

Overview Problem Set 4 (covering induction) is due on Friday Nov 8. This tutorial will allow students to practice writing proofs by induction. At this stage students will have seen simple induction (starting at any base case), induction on the evens, induction on the odds, and recursive sequences. Not all sections will have seen strong induction by this tutorial.

Suggested Activities

Activity 0. Check in with the students. How did the midterm go? How is PS4 going?

Activity 1. For each of the following facts, generalize them to statements $P(n)$. Does your n represent something in these statements (i.e. number of terms, number of sides, etc.)? For what $n \in \mathbb{N}$ is your statement true?

(1) There are $4 \cdot 3 \cdot 2 \cdot 1$ different ways to order 4 people.

- $P(n) =$ "There are $n!$ ways to order n people"
- n represents the number of people,
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(2) $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 8 \cdot 9$.

- $P(n) =$ "$2 + 4 + \ldots + 2n = n(n+1)$"
- n represents the number of terms.
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(3) $1 + \frac{7}{\pi} \leq (1 + \frac{1}{\pi})^7$.

- $P(n) =$ "$1 + \frac{n}{\pi} \leq (1 + \frac{1}{\pi})^n$"
- n represents the number of factors.
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(4) $1 + 1 + 2 + 3 + 5 + 8 + 13 = 33$.

- $P(n) =$ "$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$."
- n represents the number of terms.
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(5) $\forall x, y, z \in \mathbb{R}$ we have $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz}$.

- $P(n) =$ "$\forall x_1, \ldots, x_n \in \mathbb{R}, \dfrac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{\prod_{i=1}^{n} x_i}$".
- n represents the number of terms, or the dimension of the AMGM.

- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(6) The (inner) angles of a triangle always add up to 180 degrees. The (inner) angles of a pentagon always add up to 540 degrees.

- $P(n) =$ "The inner angles of an n-gon always add up to $180(n-2)$ degrees."
- n represents the number of sides of the polygon.
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 3$.

Here is the grading scheme for the induction questions for PS4 (this will be posted on Quercus):

- 1pt: Stated the P(n) explicitly and correctly.
- 1pt: Proved the base case explicitly and correctly. (Do not award any points if the student starts with the conclusion, and then derives a true statement.)
- 1pt: The induction hypothesis was explicitly assumed for a particular $n \in \mathbb{N}$, and its use was pointed out (correctly).
- 1pt: The structure of the proof of the inductive step was correct. (Do not award any points if the student starts with the conclusion, and then derives a true statement.)

Activity 2. Use the PS4 grading scheme to grade the following proof of the fact that "For all $n \in \mathbb{N}$, $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$."

Then, improve the proof so that it is graded 4/4.

Rough proof. Let $P(n)$ be the statement "For all $n \in \mathbb{N}$, $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$".

For $n = 1$, Note that

$$\begin{aligned} &1 + 2^1 = 2^2 - 1 \\ &\Rightarrow 1 + 2 = 3 - 1 \\ &\Rightarrow 3 = 3 \checkmark \end{aligned}$$

Now assume $P(n)$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} 1 + 2 + 4 + \ldots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1. \end{aligned}$$

Therefore $P(n+1)$ is true, as desired. □

The grading scheme says to award that proof 1/4 (the one point is for the correct structure of the inductive step). Here is the improved (4/4) proof:

Improved proof. Let $P(n)$ be the statement "$1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$".

For $n = 1$, Note that

$$1 + 2^1 = 3 = 4 - 1 = 2^{1+1} - 1.$$

Now assume $P(n)$ for some $n \in \mathbb{N}$. So

$$\begin{aligned} 1+2+4+\ldots+2^n+2^{n+1} &= 2^{n+1}-1+2^{n+1} && \text{by the IH} \\ &= 2\cdot 2^{n+1}-1 \\ &= 2^{n+2}-1. \end{aligned}$$

Therefore $P(n+1)$ is true, as desired. □

Activity 3. Let $a_1 = 1$, and let $a_{n+1} = 3a_n+1$ for $n \in \mathbb{N}$. Compute a_2, a_3, a_4 using this recursive definition. Prove that, for all $n \in \mathbb{N}$ that

$$a_n = \frac{3^n-1}{2}$$

Proof. Note that $\frac{3^1-1}{2} = \frac{2}{2} = 1 = a_1$.

Now, suppose that $a_n = \frac{3^n-1}{2}$ for a particular $n \in \mathbb{N}$.

Note that

$$\begin{aligned} a_{n+1} &= 3a_n+1 && \text{By definition} \\ &= 3\frac{3^n-1}{2}+1 && \text{By IH} \\ &= \frac{3\cdot 3^n-3+2}{2} \\ &= \frac{3^{n+1}-1}{2} \end{aligned}$$

As desired. □

Activity 4. Generalize your previous proof to find an explicit formula for the sequence defined by: $b_1 = 1$ and $b_{n+1} = 13b_n + 1$.

Line 3 of the previous proof tells us how to generalize this. The formula will be:

$$b_n = \frac{13^n-1}{12}.$$

In general, if $c \in \mathbb{R}$ and $c > 1$, then the recursive formula $a_1 = 1$ and $a_{n+1} = ca_n + 1$ will have description:

$$a_n = \frac{c^n-1}{c-1}$$

You can push stronger students to look for that formula. That fraction might look familiar as it is the geometric sum formula.

$$\frac{c^n-1}{c-1} = 1+c+c^2+\ldots+c^{n-1}.$$

This is one possible motivation for how we came up with this formula.

MAT102H5 F 2019 - Tutorial Handout - Week 9

By the end of this tutorial, students should be able to:

(1) Generalize a statement about a particular number, to a statement $P(n)$.

(2) Improve the presentation of a proof by induction.

(3) Prove that a given explicit formula generates a given recursively defined sequence.

Suggested Activities

Activity 1. For each of the following facts, generalize them to statements $P(n)$. Does your n represent something in these statements (i.e. number of terms, number of sides, etc.)? For what $n \in \mathbb{N}$ is your statement true?

(1) There are $4 \cdot 3 \cdot 2 \cdot 1$ different ways to order 4 people.

(2) $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 8 \cdot 9$.

(3) $1 + \frac{7}{\pi} \leq (1 + \frac{1}{\pi})^7$.

(4) $1 + 1 + 2 + 3 + 5 + 8 + 13 = 33$.

(5) $\forall x, y, z \in \mathbb{R}$ we have $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz}$.

(6) The (inner) angles of a triangle always add up to 180 degrees. The (inner) angles of a pentagon always add up to 540 degrees.

Activity 2. Use the PS4 grading scheme to grade the following proof (see next page) of the fact that

"For all $n \in \mathbb{N}$, $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$."

Then, improve the proof so that it is graded 4/4.

Activity 3. Let $a_1 = 1$, and let $a_{n+1} = 3a_n + 1$ for $n \in \mathbb{N}$. Compute a_2, a_3, a_4 using this recursive definition.

Prove that, for all $n \in \mathbb{N}$ that

$$a_n = \frac{3^n - 1}{2}.$$

Activity 4. Generalize your previous proof to find an explicit formula for the sequence defined by: $b_1 = 1$ and $b_{n+1} = 13b_n + 1$.

Here is the grading scheme for the induction questions for PS4 (this will be posted on Quercus):

- 1pt: Stated the $P(n)$ explicitly and correctly.
- 1pt: Proved the base case explicitly and correctly. (Do not award any points if the student starts with the conclusion, and then derives a true statement.)
- 1pt: The induction hypothesis was explicitly assumed for a particular $n \in \mathbb{N}$, and its use was pointed out (correctly).
- 1pt: The structure of the proof of the inductive step was correct. (Do not award any points if the student starts with the conclusion, and then derives a true statement.)

Proof. Let $P(n)$ be the statement "For all $n \in \mathbb{N}$, $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$".

For $n = 1$, Note that

$$\begin{aligned} &1 + 2^1 = 2^2 - 1 \\ &\Rightarrow 1 + 2 = 4 - 1 \\ &\Rightarrow 3 = 3\checkmark \end{aligned}$$

Now assume $P(n)$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} 1 + 2 + 4 + \ldots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1. \end{aligned}$$

Therefore $P(n+1)$ is true, as desired. $\square$

Tutorial for Week 10 - TA Version

By the end of this tutorial, students should be able to:

(1) State the different variations on induction. (Simple, evens, odds, different base case, strong)

(2) Decide on which version of induction to use to solve a problem.

(3) Identify a function that is not injective, and find an example why.

(4) Identify a function that is not surjective, and find an example why.

Overview Quiz 3 is coming up on the Thursday of this week. In this tutorial we will cover some of the more advanced uses/variations of induction. Last week the tutorial was more introductory in nature. This week we'll expect more from them.

The questions on injectivity and surjectivity are meant to prepare them for cardinality which is coming up soon.

Suggested Activities

Activity 0. Check in with the students. How do they feel about the upcoming Quiz 3?

Activity 1. What are the 5 different variations on induction that we looked at? State them. For each of the following facts, choose the variation that is most appropriate. (**Warning**. Not all 5 variations will show up in this activity!)

(1) Whenever n is a natural number, then $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ is an integer.

- Simple induction.

(2) $10^n - 1$ is divisible by 11 for every even natural number n.

- Induction on evens.

(3) For large enough natural numbers n, we must have $n^3 < 3^n$.

- Simple induction starting at $N = 4$.

(4) Let a_n be a sequence defined recursively as $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Given that a_1, a_2 are odd, show that all a_n are odd.

- Simple induction! The use of the word "odd" here is a distraction.

(5) Every natural number n can be written in base 10 in only one way.

- Strong induction.

Activity 2. Give specific points a, b that show that these functions are not injective:

(1) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x^2}$.

- Example: $a = 1$ and $b = -1$.

(2) $g : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $g(x) = \cos(x)$.

- Example: $a = \frac{-\pi}{4}$ and $b = \frac{\pi}{4}$.

(3) $h : \mathbb{N} \to \mathbb{R}$ given by $h(n) = (-1)^n$.

- Example: $a = 2$ and $b = 4$.

Activity 3. Give specific points y the codomain of these functions that show that these functions are not surjective:

(1) $f : \mathbb{R} \setminus \{0\} \to [0, \infty)$ given by $f(x) = \frac{1}{x^2}$.

- Example: $y = 0$, because $\frac{1}{x^2} > 0$.

(2) $g : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ given by $g(x) = \cos(x)$.

- Example: $y = 2$, because $\cos x \leq 1 < 2$ for all $x \in \mathbb{R}$.

(3) $h : \mathbb{N} \to [-1, 1]$ given by $h(n) = (-1)^n$.

- Example: $y = 0$, because h only has outputs -1 and 1.

Activity 4. Show that there are 8 different functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Proof. The functions written as $f(1)f(2)f(3)$. So for example 110 is the function $f(1) = 1, f(2) = 1, f(3) = 0$. This notation suggests the (true) fact that the collection of all functions from $\{1, 2, \dots, n\}$ to $0, 1$ is the collection of all binary strings of length n. This makes counting easier.

$$000, 001, 010, 011, 100, 101, 110, 111$$

Let them discover this fact! □

Activity 5. Show that there are 6 different <u>surjective</u> functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Proof. Everything above works EXCEPT 000 and 111. This reminds us of this important principle of counting: Instead of counting the good things, you can count the bad things. □

Activity 6. Show that there are <u>NO</u> <u>injective</u> functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Proof. This is impossible because the domain has 3 elements, but the codomain has only 2 choices. One of the choices must be repeated (by the pigeonhole principle).

Nudge them to think about how the size of the domain and the size of the codomain makes an injective function impossible. Ask them to make a more general observation. □

Activity 7. Challenge: Generalize the 3 previous exercises to functions with domain $\{1, 2, \dots, 10\}$, and codomain $\{0, 1\}$.

Proof. There are 2^n many functions, and all but 2 of them are surjective. There are never any injective functions. □

As an additional challenge, you can get them to identify a function $f : A \to \{0, 1\}$ as a subset of A. The identification is: $X = \{x \in A : f(x) = 1\}$. In this sense the function is the <u>indicator function</u> for the set A. This prove that the "the collection of all functions from A to $\{0, 1\}$" and "the collection of all subsets of A" have the same number of elements.

MAT102H5 F 2019 - Tutorial Handout - Week 10

By the end of this tutorial, students should be able to:

(1) State the different variations on induction.

(2) Decide on which version of induction to use to solve a problem.

(3) Identify a function that is not injective, and find an example why.

(4) Identify a function that is not surjective, and find an example why.

Suggested Activities

Activity 1. What are the 5 different variations on induction that we looked at? State them. For each of the following facts, choose the variation that is most appropriate. (**Warning**. Not all 5 variations will show up in this activity!)

(1) Whenever n is a natural number, then $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ is an integer.

(2) $10^n - 1$ is divisible by 11 for every even natural number n.

(3) For large enough natural numbers n, we must have $n^3 < 3^n$.

(4) Let a_n be a sequence defined recursively as $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Given that a_1, a_2 are odd, show that all a_n are odd.

(5) Every natural number n can be written in base 10 in only one way.

Activity 2. Give specific points a, b that show that these functions are not injective:

(1) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x^2}$.

(2) $g : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ given by $g(x) = \cos(x)$.

(3) $h : \mathbb{N} \to \mathbb{R}$ given by $h(n) = (-1)^n$.

Activity 3. Give specific points y the codomain of these functions that show that these functions are not surjective:

(1) $f : \mathbb{R} \setminus \{0\} \to [0, \infty)$ given by $f(x) = \frac{1}{x^2}$.

(2) $g : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ given by $g(x) = \cos(x)$.

(3) $h : \mathbb{N} \to [-1, 1]$ given by $h(n) = (-1)^n$.

Activity 4. Show that there are 8 different functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Activity 5. Show that there are 6 different surjective functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Activity 6. Show that there are NO injective functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Activity 7. Challenge: Generalize the 3 previous exercises to functions with domain $\{1, 2, \ldots, 10\}$, and codomain $\{0, 1\}$.

Tutorial for Week 11 - TA Version

By the end of this tutorial, students should be able to:

(1) Prove general facts about compositions of functions, from definitions.

(2) State and prove basic abstract results about cardinality.

Overview Quiz 4 is coming up on the Thursday of this week. It will cover sigma/pi notation, induction (again), injective/surjective compositions of functions. This tutorial will help with the Quiz.

Suggested Activities

Activity 0. Check in with the students. How did Quiz 3 go? How do they feel about the upcoming Quiz 4?

Activity 1. Let $f : A \to B$. Which of the following two statements means that f is injective? What does the other one mean? Write the contrapositives of both implications.

(1) $\forall x_1, x_2 \in A$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$.

(2) $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

1. is the definition of being a function. It is vertical line test. 2. is the definition of injective. (Its contrapositive is "2-to-2", which is a better name for 1-to-1.)

Activity 2. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. For each condition, give an example of functions $f_i : A \to B$ and $g_i : B \to A$ with the following properties, or explain why it is impossible.

(1) g_1 is the inverse of f_1.

(2) $f_2 \circ g_2(x) = g_2 \circ f_2(x)$ for all x.

(3) $f_3 \circ g_3(x) = x$ for all $x \in B$.

(4) The range of $g_4 \circ f_4$ has 2 elements, but the range of $f_4 \circ g_4$ has 1.

(5) The range of $g_5 \circ f_5$ has 3 elements, but the range of $f_5 \circ g_5$ has 2.

(1) Possible (easily).

(2) Impossible because they don't have the same domain!

(3) Possible if g_3 is the inverse of f_3.

(4) Possible: $f(1) = f(2) = 4, f(3) = 6$ and $g(4) = g(6) = 1, g(5) = 2$.

(5) Impossible since the first condition forces both functions to be bijections. (This can be seen by contradiction.)

Activity 3. For each of the following statements identify if the proof will be simple definition unwinding or not. If it is simple, summarize the proof in one sentence. These statements are for all sets A, B, C.

(1) $|A| = |A|$.

(2) $|A| = |B| \Rightarrow |B| = |A|$.

(3) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

(4) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

(5) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

(6) $|A| \leq |B|$ or $|B| \leq |A|$.

If you want, you can discuss one or two of the definition unwinding proofs.

(1) "The identity function is a bijection."

(2) "The inverse of a bijection is a bijection."

(3) "The composition of injections is an injection."

(4) "The composition of bijections is an bijection."

(5) This is the Cantor-Schroeder-Bernstein theorem. It is not obvious.

(6) This is not an obvious theorem and relies on the axiom of choice (which we do not discuss in this class).

Activity 4. In class we showed that all intervals of $\mathbb{R}$ have the same cardinality as $\mathbb{R}$.

(1) Construct a bijection that shows $|[0,1] \cup (2,3]| = |[0,2]|$.

(2) Use the Cantor-Schroder-Bernstein theorem to show that the $|[0,1] \cup (2,3]| = |[0,1] \cup [2,3]|$.

(3) Show that if I_1, I_2 are intervals of $\mathbb{R}$, then $|I_1 \cup I_2| = |\mathbb{R}|$.

(1) $f : [0,2] \to [0,1] \cup (2,3]$ given piecewise by $f(x) = x$ for $x \in [0,1]$ and $f(x) = x + 1$ for $x \in (1,2]$.

(2) The subset direction makes the identity an injection. For the other direction we can use, for example $f : [0,1] \cup [2,3] \to [0,1] \cup (2,3]$ given by $f(x) = \frac{x}{3}$.

(3) We use CSB: $I_1 \subset I_1 \cup I_2$ so $|\mathbb{R} = |I_1| \leq |I_1 \cup I_2|$. Also $I_1 \cup I_2 \subset \mathbb{R}$ so $|I_1 \cup I_2| \leq |\mathbb{R}|$. So by CSB, they have the same cardinality.

MAT102H5 F 2019 - Tutorial Handout - Week 11

By the end of this tutorial, students should be able to:

(1) Prove general facts about compositions of functions, from definitions.

(2) State and prove basic abstract results about cardinality.

Suggested Activities

Activity 1. Let $f : A \to B$. Which of the following two statements means that f is injective? What does the other one mean? Write the contrapositives of both implications.

(1) $\forall x_1, x_2 \in A$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$.

(2) $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Activity 2. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. For each condition, give an example of functions $f_i : A \to B$ and $g_i : B \to A$ with the following properties, or explain why it is impossible.

(1) g_1 is the inverse of f_1.

(2) $f_2 \circ g_2(x) = g_2 \circ f_2(x)$ for all x.

(3) $f_3 \circ g_3(x) = x$ for all $x \in B$.

(4) The range of $g_4 \circ f_4$ has 2 elements, but the range of $f_4 \circ g_4$ has 1.

(5) The range of $g_5 \circ f_5$ has 3 elements, but the range of $f_5 \circ g_5$ has 2.

Activity 3. For each of the following statements identify if the proof will be simple definition unwinding or not. If it is simple, summarize the proof in one sentence. These statements are for all sets A, B, C.

(1) $|A| = |A|$.

(2) $|A| = |B| \Rightarrow |B| = |A|$.

(3) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

(4) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

(5) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

(6) $|A| \leq |B|$ or $|B| \leq |A|$.

Activity 4. In class we showed that all intervals of $\mathbb{R}$ have the same cardinality as $\mathbb{R}$.

(1) Construct a bijection that shows $|[0, 1] \cup (2, 3]| = |[0, 2]|$.

(2) Use the Cantor-Schroder-Bernstein theorem to show that the $|[0, 1] \cup (2, 3]| = |[0, 1] \cup [2, 3]|$.

(3) Show that if I_1, I_2 are intervals of $\mathbb{R}$, then $|I_1 \cup I_2| = |\mathbb{R}|$.

Tutorial for Week 12 - TA Version

By the end of this tutorial, students should be able to:

(1) Evaluate (the truth of) mathematical statements of the forms $A \in \mathcal{P}(B)$ and $A \subseteq \mathcal{P}(B)$.

(2) Identify countable and uncountable sets.

(3) Prove that two sets have the same cardinality.

Overview By now students should have seen all of the cardinality section of the course. The material in class is mostly theoretical, so the tutorial is where they should apply their skills.

Note that the final tutorial is next week.

Suggested Activities

Activity 0. Check in with the students. How did Quiz 4 go? How are they feeling about the upcoming exam? What are they doing to prepare?

The following activity is mostly a warm-up for them to remember how power sets work. Please let them do this on t
The point of this is to remind them how power sets work.

Activity 1. Let $A = \{0, 1\}$ and $B = \{2\}$. Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$ in 3 different ways:

(1) Compute all three power sets, and compare $\mathcal{P}(A) \cup \mathcal{P}(B)$ with $\mathcal{P}(A \cup B)$.

(2) Compare their cardinalities by using the fact that for finite sets, $|\mathcal{P}(X)| = 2^{|X|}$.

(3) Prove that for any set X, $X \in \mathcal{P}(X)$. However, $A \cup B \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

The challenge of the following activity is for the students to read and use definitions involving the power set. The first 6 exercises are designed to help them unravel the definitions. Encourage them to try the first six exercises first, before you help them. Note that 1 and 4, 2 and 5, 3 and 6 are related.

Activity 2. Let $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \cup \{\text{DNE}\}$, be defined by $f(A)$ is the minimum element of A, if $A \neq \emptyset$ and $f(\emptyset) = \text{DNE}$.

Let $g : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{N})$ be the (restriction) function $g(A) = A \cap \mathbb{N}$.

(1) Compute $f(\{2, 3, 5\})$.

(2) Compute $f(\mathbb{N})$.

(3) Compute $f(\emptyset)$.

(4) Compute $f \circ g([1.5, 4) \cup [4.5, 5.5])$.

(5) Compute $f \circ g(\mathbb{R})$.

(6) Compute $f \circ g([-2, 0])$.

(7) In words, explain what the function $f \circ g$ does.

(8) Prove that if $\emptyset \neq A \subseteq B \subseteq \mathbb{R}$, then $f \circ g(A) \geq f \circ g(B)$.

Note that in MAT102, the definition of countable is "in bijection with $\mathbb{N}$", and uncountable is "infinite and not countable". Officially, in MAT102, finite sets do not count as countable sets. There is disagreement about this in the mathematical community.

Students have two main tools for identifying uncountable sets. They have seen:

(1) $\mathbb{R}$ is uncountable (and every non-trivial interval).

(2) If A is infinite, then $\mathcal{P}(A)$ is uncountable.

Activity 3. Identify which of the following sets are finite, countable, or uncountable. Give a short explanation for your choice (a complete proof is not necessary).

(1) $\{1, 2, 3, \ldots, 2019\}$.

(2) $\mathcal{Z}$.

(3) $A \cup \mathbb{R}$, where A is any set.

(4) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{0, 1\})))$.

(5) $\mathcal{P}(\mathbb{N}) \cap \mathbb{N}$.

(6) $\mathbb{Q} \cap [0, 1]$.

Activity 4. Construct a bijection that shows that the following sets are countable. Prove that your function is a bijection.

(1) $\{0\} \cup \mathbb{N}$.

(2) $\{-2019, -2018, \ldots, 1, 0\} \cup \mathbb{N}$.

(3) $\{-n : n \in \mathbb{N}\}$.

(4) $\mathbb{Z} \cup \{x + \frac{1}{2} : x \in \mathbb{Z}\}$.

(5) $\{1, 10, 100, 1000, \ldots, \}$

(6) $\mathbb{N} \times \mathbb{Q}$.

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By the end of this tutorial, students should be able to:

(1) Evaluate (the truth of) mathematical statements of the forms $A \in \mathcal{P}(B)$ and $A \subseteq \mathcal{P}(B)$.

(2) Identify countable and uncountable sets.

(3) Prove that two sets have the same cardinality.

Suggested Activities

Activity 1. Let $A = \{0, 1\}$ and $B = \{2\}$. Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$ in 3 different ways:

(1) Compute all three power sets, and compare $\mathcal{P}(A) \cup \mathcal{P}(B)$ with $\mathcal{P}(A \cup B)$.

(2) Compare their cardinalities by using the fact that for finite sets, $|\mathcal{P}(X)| = 2^{|X|}$.

(3) Prove that for any set X, $X \in \mathcal{P}(X)$. However, $A \cup B \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

Activity 2. Let $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \cup \{\text{DNE}\}$, be defined by $f(A)$ is the minimum element of A, if $A \neq \emptyset$ and $f(\emptyset) = \text{DNE}$.

Let $g : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{N})$ be the (restriction) function $g(A) = A \cap \mathbb{N}$.

(1) Compute $f(\{2, 3, 5\})$.

(2) Compute $f(\mathbb{N})$.

(3) Compute $f(\emptyset)$.

(4) Compute $f \circ g([1.5, 4) \cup [4.5, 5.5])$.

(5) Compute $f \circ g(\mathbb{R})$.

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(7) In words, explain what the function $f \circ g$ does.

(8) Prove that if $\emptyset \neq A \subseteq B \subseteq \mathbb{R}$, then $f \circ g(A) \geq f \circ g(B)$.

Activity 3. Identify which of the following sets are finite, countable, or uncountable. Give a short explanation for your choice (a complete proof is not necessary).

(1) $\{1, 2, 3, \ldots, 2019\}$.

(2) $\mathcal{Z}$.

(3) $A \cup \mathbb{R}$, where A is any set.

(4) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{0, 1\})))$.

(5) $\mathcal{P}(\mathbb{N}) \cap \mathbb{N}$.

(6) $\mathbb{Q} \cap [0, 1]$.

Activity 4. Construct a bijection that shows that the following sets are countable. Prove that your function is a bijection.

(1) $\{0\} \cup \mathbb{N}$.

(2) $\{-2019, -2018, \ldots, 1, 0\} \cup \mathbb{N}$.

(3) $\{-n : n \in \mathbb{N}\}$.

(4) $\mathbb{Z} \cup \{x + \frac{1}{2} : x \in \mathbb{Z}\}$.

(5) $\{1, 10, 100, 1000, \ldots, \}$

(6) $\mathbb{N} \times \mathbb{Q}$.

Tutorial for Week 13 - TA Version

By the end of this tutorial, students should be able to:

(1) Apply the division algorithm.

(2) Apply the Euclidean GCD algorithm.

(3) Find witnesses to Bezout's Identity.

Overview This is the last tutorial. In this tutorial the students will work through some simple GCD/Bezout's identity examples to get a feel for it. This stuff is easy for us, but challenging for them, so please let them work through these examples (avoid the temptation to present full solutions).

The final activity is just for fun. We won't test them on it, but it is more creative than the rest of the activities.

Suggested Activities

Activity 0. Check in with the students. How are they feeling about the upcoming exam? What are they doing to prepare?

Activity 1. What is the largest integer k such that $b - ka \geq 0$? What does this tell you about the q, r you get from the division algorithm so that $b = qa + r$?

(1) $a = 3$, $b = 11$.

(2) $a = 5$, $b = 53$.

(3) $a = 200$, $b = 999$.

Activity 2. Compute the GCD and LCM of the following numbers:

(1) $a = 12$, $b = 20$.

(2) $a = 10!$, $b = 2^{10}$.

(3) (**Challenge**!) $a = n!$, $b = 2^n$.

Activity 3. (1) Apply the Euclidean algorithm to find $\gcd(95, 60)$.

(2) Find integers x, y such that $95x + 60y = \gcd(95, 60)$.

(3) Are there any other integers x, y that solve the above equation?

Proof. Here is the Euclidean Algorithm:

$$\begin{aligned}
95 &= 1(60) + 35 \\
60 &= 1(35) + 25 \\
35 &= 1(25) + 10 \\
25 &= 2(10) + \boxed{5} \\
10 &= 2(5)
\end{aligned}$$

So $\gcd(95, 60) = 5$.

To find solutions we reverse this, first getting the equations:

$$\begin{aligned}35 &= 95 - 1(60)\\ 25 &= 60 - 1(35)\\ 10 &= 35 - 1(25)\\ \boxed{5} &= 25 - 2(10)\end{aligned}$$

Starting with the equation for 5, and back-substituting gives:

$$\begin{aligned}\boxed{5} &= 25 - 2(10)\\ &= (60 - 35) - 2(35 - 25)\\ &= 60 - 3(35) + 2(25)\\ &= 60 - 3(95 - 60) + 2(60 - 35)\\ &= 6(60) - 3(95) - 2(35)\\ &= 6(60) - 3(95) - 2(95 - 60)\\ &= 8(60) - 5(95).\end{aligned}$$

To get additional solutions, we "make" an additional copy of one of the terms (say, 60), and then subtract it off. For example,

$$\begin{aligned}5 &= 8(60) - 5(95)\\ &\Rightarrow\\ 65 &= (8 \cdot 13)(60) - (5 \cdot 13)(95)\\ &\Rightarrow\\ 5 &= (8 \cdot 13)(60) - 60 - (5 \cdot 13)(95)\\ &= 103(60) - 65(95)\end{aligned}$$

□

Activity 4. GCD plays an important role in many applications to number theory. If $n \in \mathbb{N}$, then $U(n) = \{x \in \mathbb{N} : \gcd(x, n) = 1 \text{ and } x \leq n\}$, and $\varphi(n) = |U(n)|$. For example, $U(10) = \{1, 3, 7, 9\}$, and $\varphi(10) = 4$.

Compute $U(n)$ and $\varphi(n)$ for the following n.

(1) $n = 6$

(2) $n = 7$

(3) $n = 8$

(4) Each n from 1 to 24. (Gather your results in a table.)

Make two conjectures about how $\varphi(n)$ works.

MAT102H5 F 2019 - Tutorial Handout - Week 13

By the end of this tutorial, students should be able to:

(1) Apply the division algorithm.

(2) Apply the Euclidean GCD algorithm.

(3) Find witnesses to Bezout's Identity.

Suggested Activities

Activity 1. What is the largest integer k such that $b - ka \geq 0$? What does this tell you about the q, r you get from the division algorithm so that $b = qa + r$?

(1) $a = 3$, $b = 11$.

(2) $a = 5$, $b = 53$.

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Activity 2. Compute the GCD and LCM of the following numbers:

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