# MAT102F - Intro. to Mathematical Proofs - Fall 2019 - UTM

# Exam Jam!

## MCQ

1. Suppose $f : A \to B$ and $g : B \to C$ are functions and $g \circ f$ is injective. Which of the following statement is always true?

   (a) $f$ is injective. (b) $g$ is injective. (c) $A = C$

2. Suppose $S$ is a statement form with $n$ variables $P_1, P_2, \ldots, P_n$, where $n \geq 3$. In exactly how many rows of the truth table for $S$ is $P_1$ false?

   (a) 1. (b) 2. (c) $2^{n-1}$. (d) $2^n$.

3. Suppose $S$ is a statement form with $n$ variables $P_1, P_2, \ldots, P_n$, where $n \geq 3$. In exactly how many rows of the truth table for $S$ is $P_1$ false while $P_2$ is true?

   (a) 1. (b) 2. (c) $2^{n-1}$. (d) $2^{n-2}$.

4. Which statement is a tautology?

   (a) $P \implies (P \implies Q)$
   (b) $(P \wedge Q) \implies (P \vee R)$
   (c) $(P \implies Q) \iff Q$
   (d) $(P \implies Q) \implies Q$

5. Which of the following statement is always true?

   (a) $\{1, 2, 3\} = \{2, 1, 3, 3, 2\}$
   (b) $\{5, \emptyset\} = \{5\}$
   (c) $\{5\} \in \{2, 5\}$
   (d) $\{1, 2\} \subseteq \{3, \{1, 2\}\}$

6. Given two sets $A$ and $B$, which of the following statement is always true?

   (a) $P(A \cup B) = P(A) \cup P(B)$.
   (b) $\emptyset \notin P(A) \setminus P(B)$.
   (c) $P(A \cap B) \subseteq P(A) \cap P(B)$.
   (d) $P(A \setminus B) \neq P(A)$.

7. How many zeroes would appear on the right side of the number $N = 100!$?

   (a) 24 (b) 25 (c) 26 (d) 20

8. Given $|S| = m$ and $|T| = n$ where $S$ and $T$ are two sets, how many functions are there with domain $S$ and codomain $T$?

   (a) $n^m$ (b) $m^n$ (c) $mn$ (d) $m+n$

## Long Answers

1. Prove that every integer $n$ satisfies the congruence $n^3 \equiv n \pmod 6$.

2. Suppose that $f : \{1, 2, \ldots, 2019\} \to \{0, 1\}$ is a function such that $f(1) = 0$ and $f(2019) = 1$. Prove that there is an $k \in \{1, 2, \ldots, 2018\}$ such that $f(k) = 0$ and $f(k+1) = 1$.

3. Let $A$ be a set.

   (a) Prove that $R = A \times A$ is an equivalence relation on $A$.

   (b) Prove that $S = \{(a, a) : a \in A\}$ is an equivalence relation on $A$.

   (c) For the above relations, is $R \subseteq S$, $S \subseteq R$ or neither?

4. Given $(k, m) = 1$, show that the following cancellation law holds:

$$ka \equiv kb \pmod m \implies a \equiv b \pmod m$$

   Use this result, prove every integer is congruent modulo $m$ to exactly one of the integers

$$0, k, 2k, \ldots, (m-1)k.$$

5. Show that the set $\mathbb{N} \times \mathbb{N}$ can be expressed as the union of a countably infinite family of countably infinite sets.

6. Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ :

$$g(m, n) = 2^{m-1}(2n-1)$$

   Show that $g$ is a bijection. What does it say about the cardinality of $\mathbb{N} \times \mathbb{N}$?

7. A set $A$ is <u>defined</u> to be "Dedekind infinite" if there exists an injection $f : A \to A$ that is not a surjection.

   (a) Prove that $\mathbb{N}$ and $\mathbb{R}$ are Dedekind infinite.

   (b) Prove that $\{1\}$ is not Dedekind infinite.

   (c) (**Challenge!**) Prove by induction that for all $n \in \mathbb{N}$ that $\{1, \ldots, n\}$ is not Dedekind infinite.