MAT102H5 F - FALL 2019 - PROBLEM SET 1 - SOLUTIONS

Submission

- **You must submit your completed problem set on Quercus by 5:00pm Friday Sept 20, 2019**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If possible, please submit your assignment as a PDF, or as a clear picture.
- If you require additional space, please insert extra pages.

Additional Instructions

You must justify and support your solution to each question.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

I, ____, affirm that this assignment represents entirely my own
STUDENT NAME
efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters.*

By signing this form I agree that the statements above are true.

If I do not agree with the statements above, I will not submit my assignment and will consult the course coordinator (Professor Mike Pawliuk) immediately.

Student Name: ____

Student Number: ____

Signature: ____

Date: ____

Often there are multiple different ways to solve the same problem.

Problem 1. Suppose that $p(x) = ax^2 + bx + c$ is a quadratic polynomial. Show, **by using the quadratic formula** that $p(x)$ and the transformed polynomial $g(x) = p(x-2)$ have the same number of roots.
For example, the polynomial $p(x) = x^2 + 1$ has no roots, and the polynomial $g(x) = p(x-2) = (x-2)^2 + 1$ also has no roots.

Solution. We will show that $p(x)$ and $g(x)$ have the same discriminant, thus they must have the same number of roots by the quadratic formula.
The discriminant of $p(x)$ is $b^2 - 4ac$. Now

$$\begin{aligned} g(x) &= p(x-2) \\ &= a(x-2)^2 + b(x-2) + c \\ &= (a)x^2 + (-4a+b)x + (4a-2b+c). \end{aligned}$$

So the discriminant of $g(x)$ is

$$(-4a+b)^2 - 4(a)(4a-2b+c) = 16a^2 - 8ab + b^2 - 16a^2 + 8ab - 4ac = b^2 - 4ac,$$

which is the same as the discriminant of $p(x)$.

Grading. [2 points total]. 1 point for correctly using the discriminant, 1 point for a readable explanation/solution. Be generous with the point for explanation; basically, did they put in an effort to be clear.
Other solutions are possible, and acceptable, so long as the quadratic formula was used at some point. In particular, breaking up the quadratic into cases based on the number of roots is acceptable (but completely unnecessary).

Problem 2. Suppose that $s(x) = ax^3 + bx^2 + cx + d$ is a cubic polynomial. Show, that $s(x)$ and the transformed polynomial $t(x) = s(x-2)$ have the same number of roots. (Note that you cannot apply the quadratic formula here.)

Solution. We will show that *any* horizontal shift preserves the number of roots of $s(x)$. Let $k \in \mathbb{R}$ which we will use as our shift. Define $t(x) = s(x-k)$. Let S be the number of roots of $s(x)$, and let T be the number of roots of $t(x)$.
First, we show that $t(x)$ has at least as many roots as $s(x)$, possibly more i.e. $S \leq T$.
Suppose that r is a root of $s(x)$. So by definition, $s(r) = 0$. Note that $r + k$ is a root of $t(x)$ because

$$t(r+k) = s(r+k-k) = s(r) = 0.$$

Notice that if $r_1 \neq r_2$ then $r_1 + k \neq r_2 + k$, so all the different roots of $s(x)$ get sent to different roots of $t(x)$. So we have shown that $S \leq T$. Now we need to argue why $t(x)$ doesn't have any **more** roots than $s(x)$.
Notice that

$$t(x - (-k)) = s(x + k - k) = s(x),$$

or in other words, we see that $s(x)$ is a horizontal shift of $t(x)$. So our previous argument shows that $T \leq S$. So we must conclude that $S = T$. In other words, $s(x)$ and $t(x)$ have the same number of roots.

Grading. See next question for grading scheme.

Problem 3. Suppose that $p(x) = ax^2 + bx + c$ is a quadratic polynomial. Show, **without using the quadratic formula** that $p(x)$ and the transformed polynomial $g(x) = p(x - 2)$ have the same number of roots. (Adapt your solution from Problem 2.)

Solution. Literally copy/paste the solution for 2. There we made no mention of the fact that $s(x)$ was a cubic; it works for all polynomials.

Grading. [Problem 2 and 3: 5 points total] These two questions should be graded together, as they are really the same question.

(1) 2 points: Recognizing that the roots of $s(x)$, shifted 2 units, are roots of $t(x)$. (In a perfect world they would argue that none of these new, shifted roots are the same, but that's not required for grades.)
(2) 1 point: Any observation or argument why $t(x)$ doesn't have **more** roots. (They don't need to prove that **every** horizontal shift preserves the number of roots, just for shifting by $+2$ or -2).
(3) 1 point: Giving a solution for Problem 3 that (1) adapts their solution for Problem 2, (2) applies to quadratics, and (3) doesn't use discriminants or the quadratic formula.
(4) 1 point: An attempt was made to explain their solutions. (Be generous with this point,)

Other solutions are possible, and acceptable. In particular, breaking up the cubic into cases based on the number of roots is acceptable (but ultimately will come back to this argument about shifts). Remove a point if they consider cases, but do not consider **all** possible forms of a cubic.
A solution that uses the cubic formula is technically possible, but is very time consuming.

Problem 4 (You do not need to submit a solution to this question). Reflect on which of your solutions (to Problem 1 or Problem 3) you prefer. Why do you prefer that one?

Problem 5. (1.5.27.c from textbook) Find all the integers that divide -20. Explain your answer briefly. (How do you know your list is complete?)

Solution. The divisors are $-20, -10, -5, -4, -2, -1, 1, 2, 4, 5, 10, 20$.
Why is this all of them? There are two cases to consider:

(1) integers in $[-20, 20]$ that are not on our list, and
(2) integers smaller than -20 or larger than 20.

The first case can be verified by tediously checking each missing number, or by observing that $-20 = (-1)2^2 \cdot 5$, so the only divisors of -20 must be products of those factors.
The second fact is because if $d| - 20$, then $|d| \leq 20$. This is because, if $|d| > 20$ and $n \in \mathbb{Z}$, then $|dn| = |d||n|$ is either 0, or larger than 20 (since the smallest non-zero value of $|n|$ is 1.)

Grading. [2 points] 1 point for a complete list, 1 point for any explanation whatsoever.

Problem 6. (1.5.30. from textbook) Show that $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} - 2\sqrt{6}$ is a rational number.

Solution. The main tool is multiplying by the conjugate. Note that

$$\begin{aligned} \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} - 2\sqrt{6} &= \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} - 2\sqrt{6} \\ &= \frac{3+2+2\sqrt{3}\sqrt{2}}{3-2} - 2\sqrt{6} \\ &= 5 + 2\sqrt{6} - 2\sqrt{6} \\ &= 5 = \frac{5}{1}. \end{aligned}$$

Thus it is a rational number.

Grading. [2 points] Award two points if the solution is mostly correct. Saying that 5 is rational, without comment, is acceptable.

Problem 7 (You do not need to submit a solution to this question). Generalize Question 6. That means find other ways that you can replace $3, 2$ and 6 in the above number and still get something rational. Generalizing a statement shows us what was important about the previous number and what relationship of the numbers $3, 2$ and 6 actually mattered.

Problem 8. (3.7.3 from textbook) Consider the following statement: "The equation $x^2 + y^2 = 1$ has a solution (x, y) in which both x and y are both natural numbers". Express the statement using the logic symbols, decide if the statement is true or false, and explain your decision briefly.

Solution. The statement is:

$$(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x^2 + y^2 = 1]$$

or, equivalently,

$$(\exists x, y \in \mathbb{N})[x^2 + y^2 = 1].$$

This statement is false. To see that, note that the smallest value of natural numbers x, y is 1. In other words, $1 \leq x$ and $1 \leq y$. Also, we must have $1 \leq x^2$ and $1 \leq y^2$, and so

$$x^2 + y^2 \geq 1 + 1 = 2 > 1.$$

Grading. [3 points] 1 point for a correct statement, 1 point for asserting that it is false, and 1 point for a reasonable argument.
Be generous with what "reasonable argument" means since we have not covered inequalities yet in this course.

Problem 9. (3.7.8.c from textbook) Construct the truth table for $(P \Rightarrow Q) \Rightarrow P$.
Suggestion: If you are planning to make CS PoST, solve this problem by writing code that takes as an input `"(P=>Q)=>P"` and prints out the truth table. Your submitted solution should be only the printed truth table.

Solution. Here is the truth table. Note that $(P \Rightarrow Q) \Rightarrow P$ is logically equivalent to P.

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

Grading. [2 points] 2 points for a completely correct solution, and only 1 point for a solution with an error. Only the final column is required.