MAT102H5 F - FALL 2019 - PROBLEM SET 3 - SOLUTIONS

Submission

- **You must submit your completed problem set on Quercus by 5:00pm Friday October 25, 2019**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If possible, please submit your assignment as a PDF, or as a clear picture.
- If you require additional space, please insert extra pages.
- **You must include a signed and completed version of this cover page**.

Additional Instructions

You must justify and support your solution to each question.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

I, ______________________________ (STUDENT NAME), affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters.*

By signing this form I agree that the statements above are true.

If I do not agree with the statements above, I will not submit my assignment and will consult the course coordinator (Professor Mike Pawliuk) immediately.

Student Name: ______________________________

Student Number: ______________________________

Signature: ______________________________

Date: ______________________________

Recall that two fractions $\frac{p}{q}, \frac{a}{b}$ are equivalent if $pb = qa$. For example, $\frac{3}{6}$ is equivalent to $\frac{4}{8}$ because $3\cdot 8 = 24 = 6\cdot 4$. Also, recall that a transversal is a set of elements such that no two elements are equivalent.

Consider the list:

$$\frac{1}{2}, \frac{2}{1}, \frac{0}{1}, \frac{10}{5}, \frac{5}{2}, \frac{2}{5}, \frac{0}{2}, \frac{1}{1}, \frac{-10}{-5}, \frac{5}{10}, \frac{4}{2}, \frac{-2}{5}$$

Problem 1. What is the number of elements of the largest transversal subset of that list? Briefly explain how you know your answer is correct.
Optional: If you are trying to make CS Post, this question should be too easy for you. Instead, solve the same question but with the list `ps3.txt`, posted on Quercus, which contains 1000 fractions. Briefly explain how you solved this question (but you do not need to submit any code).

Problem 2. What is the number of elements of the largest equivalence class that is a subset of that list? Briefly explain how you know your answer is correct.
Optional: If you are trying to make CS Post, this question should be too easy for you. Instead, solve the same question but with the list `ps3.txt`, posted on Quercus, which contains 1000 fractions. Briefly explain how you solved this question (but you do not need to submit any code).

Solution. For questions 1 and 2 it is helpful to sort the list into its equivalence classes:

- $\frac{2}{1}, \frac{10}{5}, \frac{-10}{-5}, \frac{4}{2}$,
- $\frac{1}{2}, \frac{5}{10}$,
- $\frac{0}{1}, \frac{0}{2}$,
- $\frac{5}{2}$,
- $\frac{2}{5}$,
- $\frac{1}{1}$,
- $\frac{-2}{5}$

From this we see that the largest equivalence class has 4 elements, and the largest transversal has 7 elements.

Grading. Each question is worth two points: 1 pt for the correct answer (only the number is needed) and 1pt for some sort of reasoning/explanation (such as sorting them by equivalence classes).

Solution. I've attached my sample Python code. For both questions you build the equivalence classes in a greedy way:

(1) Add the first element of the list to its own equivalence class.
(2) For each element on the list, check to see if it is equivalent to anything from any of your previously built equivalence classes (you only need to check one element from each class).
(3) If it is equivalent to something, then include it in that equivalence class.
(4) Otherwise, make a new class with only that one element.

This runs in $O(n^2)$ time where n is the length of the list of fractions. The worst case is when your starting list is already a transversal, in which case you'll make $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ many comparisons.
It turns out the largest transversal has 841 elements, and there are many equivalence classes with 8 elements.

Grading. If the student attempted the more challenging problem, and their solution is at all reasonable, then award the full 4 marks. You may remove marks if you cannot understand their solution, or if their numbers are way off.

Problem 3. (7.5.13 from textbook) On the set $\mathbb{N} \times \mathbb{N}$, define the following relation:

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c.$$

(1) Show that this is an equivalence relation.
(2) Describe the equivalence class of $(1, 1)$.

Solution. **Part a**.
Reflexive. Let $(a, b) \in \mathbb{N} \times \mathbb{N}$. Note that $a + b = b + a$, so $(a, b) \sim (a, b)$.
Symmetric. Suppose $(a, b) \sim (c, d)$. So $a + d = b + c$. Therefore, $c + b = d + a$. So $(c, d) \sim (a, b)$.
Transitive. Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Therefore, $a + d = b + c$ and $c + f = d + e$. Adding both together gives:

$$a + d + c + f = b + c + d + e.$$

Cancelling gives $a + f = b + e$. Therefore $(a, b) \sim (e, f)$.
Part b. By definition, $[(1, 1)] = \{(a, b) : (1, 1) \sim (a, b)\}$. We see that $(1, 1) \sim (a, b)$ if and only if $1 + b = a + 1$. In other words, $a = b$.
So $[(1, 1)] = \{(x, x) : x \in \mathbb{N} \times \mathbb{N}\}$.

Grading. Part a is worth 3 points: 1 pt for each property. Part b is worth 1 point for a correct answer.

Problem 4. (1.5.6.a from textbook) Solve the inequality $\frac{2}{x} > 3x$.

Solution. Case 1. Assume $x > 0$.

$$\begin{aligned} \frac{2}{x} > 3x &\Leftrightarrow 2 > 3x^2 \\ &\Leftrightarrow \frac{2}{3} > x^2 \\ &\Leftrightarrow 0 < x < \sqrt{\frac{2}{3}} \end{aligned}$$

Case 2, Assume $x < 0$.

$$\begin{aligned} \frac{2}{x} > 3x &\Leftrightarrow 2 < 3x^2 \\ &\Leftrightarrow \frac{2}{3} < x^2 \\ &\Leftrightarrow x < -\sqrt{\frac{2}{3}} \end{aligned}$$

Together this means that the set of solutions to the inequality is $(-\infty, -\sqrt{\frac{2}{3}}) \cup (0, \sqrt{\frac{2}{3}})$.

Grading. This question is worth 2 points. 1 point for breaking it up into cases, 1 point for the correct solution set.

Problem 5. (1.5.18 from textbook) Show that for any two positive real numbers a, b:

$$\frac{a}{a+2b} + \frac{b}{b+2a} \geq \frac{1}{2}.$$

(Do not present a backwards "proof" that starts from the conclusion, and derives something true. Start from something true, and eventually conclude this inequality.)

Solution. We start from the inequality

$$ab \leq 2a^2 + 2b^2.$$

This is true because, if $a \leq b$ then $ab \leq b^2 \leq 2b^2$, and if $b \leq a$ then $ab \leq a^2 \leq 2a^2$. In both cases, since $ab \leq 2a^2 + 2b^2$.

From now on, note that $a, b, 2a+b, a+2b$ are all positive, so we may multiply inequalities by those numbers without changing the inequality. Now from that inequality we have:

$$\begin{aligned} ab \leq 2a^2 + 2b^2 &\Rightarrow 5ab + 2a^2 + 2b^2 \leq 4a^2 + 4b^2 + 4ab \qquad \text{Adding } 4ab + 2a^2 + 2b^2 \text{ to both sides} \\ &\Rightarrow \frac{(a+2b)(b+2a)}{2} \leq 2a^2 + 2b^2 + 2ab \\ &\Rightarrow \frac{1}{2} \leq \frac{a(b+2a) + b(a+2b)}{(a+2b)(b+2a)} \\ &\Rightarrow \frac{1}{2} \leq \frac{a(b+2a)}{(a+2b)(b+2a)} + \frac{b(a+2b)}{(a+2b)(b+2a)} \\ &\Rightarrow \frac{1}{2} \leq \frac{a}{(a+2b)} + \frac{b}{(b+2a)} \end{aligned}$$

As desired.

Grading. This question is worth 3 points. 1pt for proving $ab \leq 2a^2 + 2b^2$, 1 pt the correct structure of the proof, and 1 pt for correct algebra.

Problem 6. Prove that $(\forall n \in \mathbb{N})(\forall x \in \mathbb{Z})[x^{2n}$ is congruent to 0 or 1 modulo 4].

Solution. Throughout this question $\cong$ will always be congruence modulo 4.
Let $x \in \mathbb{Z}$, and let $n \in \mathbb{N}$. If x is even, then there is an integer k such that $x = 2k$. Then:

$$x^{2n} = \left(x^2\right)^n = \left(4k^2\right)^n \cong 0^n \cong 0.$$

If x is odd, then there is an integer k such that $x = 2k+1$. Then:

$$x^{2n} = \left(x^2\right)^n = \left(4k^2+4k+1\right)^n \cong (0+0+1)^n \cong 1^n \cong 1.$$

Grading. This is worth 2 points: 1 point for a correct argument, and 1 point for a clear write up (be generous with this point).

Problem 7. Use the previous problem to prove that

$$-1 + 4x + x^2 + 8x^3 + x^4 = 0$$

has no integer solutions.

Solution. Throughout this question $\cong$ will always be congruence modulo 4.
Note that

$$-1 + 4x + x^2 + 8x^3 + x^4 \cong -1 + 0 + x^2 + 0 + x^4 \cong -1 + x^2 + x^4$$

If x is even, then by the previous question, $-1 + x^2 + x^4 \cong -1 + 0 + 0 \cong -1 \ncong 0$. So this has no even solutions.
If x is odd, then by the previous question, $-1 + x^2 + x^4 \cong -1 + 1 + 1 \cong 1 \ncong 0$. So this has no odd solutions.
If it has no even solutions, and no odd solutions, then it has no integer solutions.

Grading. This Q is worth 2 points. 1 point for a correct argument, and 1 point for a clear write up (be generous with this point). Remove a point if the result from Q7 was not used.