MAT102H5 F - FALL 2019 - PROBLEM SET 5 - SOLUTIONS

Submission

- **You must submit your completed problem set on Quercus by 5:00pm Friday November 29, 2019**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If possible, please submit your assignment as a PDF, or as a clear picture.
- If you require additional space, please insert extra pages.
- **You must include a signed and completed version of this cover page**.

Problem 1. Let $A \subseteq \mathbb{R}$ be a countable set. Prove that $\mathbb{R} \setminus A$ is uncountable.

Solution. **Standard Solution**. We first prove a useful lemma:
Lemma. If X, Y are disjoint countable (or finite) sets, then $|X \cup Y| \leq |\mathbb{N}|$.
Proof. By definition, there are injections $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$. Let $h : X \cup Y \to \mathbb{Z}$ be defined by $h(x) = f(x)$ for all $x \in X$, and $h(y) = -g(y)$ for all $y \in Y$. Since X and Y are disjoint, this function is well-defined. Once we prove that h is an injection, we will know that $|X \cup Y| \leq |\mathbb{Z}| = |\mathbb{N}|$.

To prove h is an injection, assume that $h(a) = h(b)$, with $a, b \in X \cup Y$. If $a, b \in X$ then $f(a) = h(a) = h(b) = f(b)$ and since f is an injection, we must have $a = b$. If $a, b \in Y$, then $-g(a) = h(a) = h(b) = -g(b)$. So $g(a) = g(b)$, and since g is an injection then $a = b$. Finally, $a \in X, b \in Y$ cannot happen since $h(a) = f(a) > 0 > h(b) = -g(b)$

Rest of solution. We know that $|\mathbb{N}| < |\mathbb{R}|$ from class, and that $(\mathbb{R} \setminus A) \cup A = \mathbb{R}$.

Suppose for the sake of contradiction that $\mathbb{R} \setminus A$ is not uncountable. So either $\mathbb{R} \setminus A$ is finite, or countable. In either case the lemma (which applies because $\mathbb{R} \setminus A$ and A are disjoint) tells us that $|\mathbb{R}| = |(\mathbb{R} \setminus A) \cup A| \leq |\mathbb{N}|$, which is a contradiction.

Solution. **Diagonalization proof**. There is a proof which uses Cantor's diagonalization argument. I will only sketch the proof here (since a more formal proof is above).

Suppose for the sake of contradiction that $\mathbb{R} \setminus A$ is not uncountable. So we may list out the elements as $x_1, x_2, \ldots,$ (possibly repeating the final x_n infinitely if the list is finite). Since A is countable, we may list the elements of A as $a_1, a_2, \ldots,$.

Now apply Cantor's Diagonalization argument to the list $x_1, a_1, x_2, a_2, \ldots$. This will produce a real number x that is neither in $\mathbb{R} \setminus A$ nor in A. This is a contradiction.

Grading. 3 points total. Any correct argument of the fact that the union of two at most countable sets is countable is worth 2 points: 1 point for using definitions accurately and correctly, 1 point for the "weaving" idea (or using $\mathbb{Z}$). The rest of the argument, done correctly, is worth 1 point.
Do not worry about the subtlety that "not uncountable" means "countable or finite".

The next two problems together will prove "$A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$".

Problem 2. Let A, B be sets. Prove that if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Explain why we can conclude that if $A = B$, then $\mathcal{P}(A) = \mathcal{P}(B)$.

Solution. Assume $A \subseteq B$. Note

$$\begin{aligned} X \in \mathcal{P}(A) &\Rightarrow X \subseteq A && \text{By definition of } \mathcal{P}(A) \\ &\Rightarrow X \subseteq B && \text{since } A \subseteq B \\ &\Rightarrow X \in \mathcal{P}(B) && \text{By definition of } \mathcal{P}(B) \end{aligned}$$

By exchanging the roles of A and B and repeating the argument we get $B \subseteq A \Rightarrow \mathcal{P}(B) \subseteq \mathcal{P}(A)$. So, using the double subset definition of set equality, we can conclude that if $A = B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Grading. 2 points total. 1 point for the argument (using the definitions correctly), and 1 point for correctly concluding the more general statement.

Problem 3. Let A, B be sets. Prove that if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$. Explain why we can conclude that if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

Solution. Assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Note

$$\begin{aligned} x \in A &\Rightarrow \{x\} \subseteq A \\ &\Rightarrow \{x\} \in \mathcal{P}(A) \qquad \text{By definition of } \mathcal{P}(A) \\ &\Rightarrow \{x\} \in \mathcal{P}(B) \qquad \text{since } \mathcal{P}(A) \subseteq \mathcal{P}(B) \\ &\Rightarrow \{x\} \subseteq B \qquad \text{By definition of } \mathcal{P}(B) \\ &\Rightarrow x \in B \end{aligned}$$

By exchanging the roles of A and B and repeating the argument we get $\mathcal{P}(B) \subseteq \mathcal{P}(A) \Rightarrow B \subseteq A$. So, using the double subset definition of set equality, we can conclude that if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A = B$.

Grading. 2 points total. 1 point for the argument (using the definitions correctly), and 1 point for correctly concluding the more general statement.

The next three problems are all related.

Problem 4. Let A be a set. Define B to be the collection of all functions $f : \{1\} \to A$. Prove that $|A| = |B|$ by constructing a bijection $F : A \to B$.

Solution. For $a \in A$, let $f_a : \{1\} \to A$ be the function defined by $f_a(1) = a$. Define $F : A \to B$ by $F(a) = f_a$.

Claim 1. F is an injection.

Suppose $a, b \in A$ such that $F(a) = F(b)$. So $f_a = f_b$. Thus

$$a = f_a(1) = f_b(1) = b.$$

So $a = b$.

Claim 2. F is a surjection.

Let $g \in B$. Let $a = g(1)$. Note that $F(a) = f_a$ is the function $f_a : \{1\} \to A$ such that

$$f_a(1) = a = g(1).$$

So $g = f_a = F(a)$.

Grading. 3 points total: 1 point for an accurate and precise definition of F, 1 point each for surjective and injective.
Remove at most 1 point for sloppy writing that is otherwise correct, but be generous.

Problem 5. Let A be a set. Define C to be the collection of all functions $f : \{0,1\} \to A$. Prove that $|A \times A| = |C|$ by constructing a bijection $F : A \times A \to C$.

Solution. For $a, b \in A$, let $f_{a,b} : \{0,1\} \to A$ be the function defined by $f_{a,b}(0) = a$ and $f_{a,b}(1) = b$. Define $F : A \times A \to C$ by $F(a,b) = f_{a,b}$.

Claim 1. F is an injection.

Suppose $a, b, c, d \in A$ such that $F(a,b) = F(c,d)$. So $f_{a,b} = f_{c,d}$. Thus

$$a = f_{a,b}(0) = f_{c,d}(0) = c.$$

So $a = c$. Also

$$b = f_{a,b}(1) = f_{c,d}(1) = d.$$

So $b = d$. Thus $(a,b) = (c,d)$.

Claim 2. F is a surjection.

Let $g \in C$. Let $a = g(0)$ and $b = g(1)$. Note that $F(a,b) = f_{a,b}$ is the function $f_{a,b} : \{0,1\} \to A$ such that

$$f_{a,b}(0) = a = g(0),$$

and

$$f_{a,b}(1) = b = g(1).$$

So $g = f_{a,b} = F(a,b)$.

Grading. 3 points total: 1 point for an accurate and precise definition of F, 1 point each for surjective and injective.
Remove at most 1 point for sloppy writing that is otherwise correct, but be generous.

Problem 6. Let A be a set. Define D to be the collection of all functions $f : A \to \{0,1\}$. Prove that $|\mathcal{P}(A)| = |D|$ by constructing a bijection $F : \mathcal{P}(A) \to D$.

Solution. For $X \subseteq A$, let $f_X : A \to \{0,1\}$ be the function defined by $f_X(x) = 0$ if $x \notin X$ and $f_X(x) = 1$ if $x \in X$. Define $F : \mathcal{P}(A) \to D$ by $F(X) = f_X$.

Claim 1. F is an injection.

Suppose $X, Y \in \mathcal{P}(A)$ such that $F(X) = F(Y)$. So $f_X = f_Y$. To prove $X = Y$ we use double subset.

Let $x \in X$. So $1 = f_X(x) = f_Y(x)$. So $x \in Y$. If $y \in Y$, then $1 = f_Y(y) = f_X(y)$. So $y \in X$. Thus $X = Y$.

Claim 2. F is a surjection.

Let $g \in D$. Let $X = \{x \in A : g(x) = 1\} \subseteq A$. In other words, $X \in \mathcal{P}(A)$. Note that $F(X) = f_X$ is the function $f_X : A \to \{0,1\}$ such that $f_X(x) = 1 = g(x)$ for all $x \in A$, and $f_X(x) = 0 = g(x)$ for all $x \notin A$. So $g = f_X = F(X)$.

Grading. 3 points total: 1 point for an accurate and precise definition of F, 1 point each for surjective and injective.
Remove at most 1 point for sloppy writing that is otherwise correct, but be generous.

In the following two problems we will refer to Cantor's Diagonalization Argument (as presented in pages 124 and 125 of the textbook). For these two questions we will replace the definition of a_k in the textbook with:

$$a_k = \begin{cases} 1 & \text{if the } k\text{th digit of } f(k) \text{ is } 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 7. You are a consultant for a friend designing a new video-game. Every player in the game is assigned a unique ID which is a binary string with 3 digits. Currently only the following three IDs have been assigned to players: $000, 001$ and 010.
Apply Cantor's Diagonalization argument to get an ID for a 4th player that is different from the three IDs already used.

Solution. Applying Cantor's Diagonalization argument to:

$$\boxed{0}00$$
$$0\boxed{0}1$$
$$01\boxed{0}$$

gives an ID of 111.

Grading. 1 point, all or nothing.

Problem 8. (Continuing Problem 7) Show that from the four IDs (the three originals and the new diagonalized one) you can choose three IDs I_1, I_2, I_3 so that they will generate a 5th new ID when Cantor's Diagonalization Argument is applied to I_1, I_2, and I_3.
By repeatedly doing this, show that you can generate all 8 possible IDs.

Solution. We need to generate $011, 100, 101$, and 110. Here are those methods:

$$\boxed{1}11$$
$$0\boxed{0}1$$
$$00\boxed{0}$$

gives an ID of 011.

$$\boxed{0}00$$
$$1\boxed{1}1$$
$$01\boxed{1}$$

gives an ID of 100.

$$\boxed{0}00$$
$$1\boxed{1}1$$
$$01\boxed{0}$$

gives an ID of 101.

$$\boxed{0}00$$
$$0\boxed{0}1$$
$$01\boxed{1}$$

gives an ID of 110.

Grading. 2 points total: 2 if it looks more or less correct. Do not look at this question carefully.