THE UNIVERSITY OF TORONTO - MISSISSAUGA - Quiz 1 - V1 - SOLUTIONS
MAT102H5F - Fall 2019 - LEC0101-LEC0107
Time: 45 minutes
Date: Thursday September 19, 2019. 7:10PM - 7:55PM.
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Aids: None.

Instructions:

- Do not write on the QR code at the top of each page.
- Only the **front** of each page will be graded. (You may use the **backs** of pages for rough work.)
- You may use Page 5 for any additional work you want graded. If you use this page, please indicate that on the original page containing the question.
- Do not remove any pages.
- Your answers to the multiple choice questions must be recorded on the final page of the booklet.
- The quiz is out of 20 points.

Instructions: There are five (5) multiple choice questions worth two (2) points each. Choose the answer that is most correct. Your answers must be recorded on the final page of the booklet.

MC1 (2pts) Reese conjectures that: "$\sqrt{n}$ is irrational for every natural number $n > 1$". Her conjecture is:

A. False, because $\sqrt{2}$ is rational.
B. True, because $\sqrt{2}$ is irrational.
C. False, because $\sqrt{4}$ is rational. $\longleftarrow$ [**Correct**.]
D. False, because $\sqrt{-1}$ is undefined.
E. False, because $\sqrt{0}$ is rational.

MC2 (2pts) Is 0 an even integer?

A. Yes, because $2 \cdot 0 = 0$. $\longleftarrow$ [**Correct**.]
B. No, because 0 cannot be divided into two equal parts.
C. No, because only positive integers can be even.
D. Yes, because it is defined to be 0.
E. Yes, because $0 + 0 = 0$.

MC3 (2pts) What are **all** the integers that divide 6?

A. $1, 2, 3, 4, 5, 6$.
B. $-6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6$.
C. $1, 2, 3, 6$.
D. $-6, -3, -2, -1, 1, 2, 3, 6$. $\longleftarrow$ [**Correct**.]
E. $2, 3$.

MC4 (2pts) How many different roots does $100x^2 + x + 100$ have?

A. 0 $\longleftarrow$ [**Correct**.]
B. 1
C. 2
D. 3
E. 4

MC5 (2pts) You are told that 2 divides a and 3 divides b, and a, b are integers. What can you conclude?

A. a must be larger than b.
B. a must be smaller than b.
C. $|a|$ must be larger than $|b|$.
D. $|a|$ must be smaller than $|b|$.
E. You can't conclude anything. $\longleftarrow$ [**Correct**.]

Instructions: There are two (2) long answer questions worth five (5) points each, with multiple parts. Provide a complete solution, with justification, in the space provided.

Q1.1 (1 POINT)

Complete the following definition: "A number x is rational if ..."

Solution. $(\exists a \in \mathbb{Z})(\exists b \in \mathbb{N}) \left[x = \frac{a}{b}\right]$

Grading. All or nothing (but round up to 1 pt). They can give their solution in words if they like; they don't need to use only logic symbols. Also accept $x = \frac{a}{b}$ with $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{Z}$.

Q1.2 (2 POINTS)

Omar claims that "$\frac{p}{q} + \frac{a}{b} = \frac{p+a}{q+b}$ for all rational numbers $\frac{p}{q}, \frac{a}{b}$". Prove his conjecture, or provide a counterexample.

Solution. This conjecture is false. For example let $\frac{p}{q} = \frac{1}{2} = \frac{a}{b}$. Thus:

$$\frac{p}{q} + \frac{a}{b} = \frac{1}{2} + \frac{1}{2} = 1 \neq \frac{1}{2} = \frac{2}{4} = \frac{p+a}{q+b}$$

Grading. 1 pt for saying it is false, 1 pt for a correct counterexample. No reasoning needed.

Q1.3 (2 POINTS)

Daria conjectures that "If you add three rational numbers together you always get a rational number". Prove her conjecture, or provide a counterexample.

Solution. This is true. Let $\frac{a}{b}, \frac{c}{d}, \frac{s}{t}$ be rationals. That is $a, c, s \in \mathbb{Z}$ and $b, d, t \in \mathbb{N}$. Note that

$$\frac{a}{b} + \frac{c}{d} + \frac{s}{t} = \frac{adt + cbt + sbd}{bdt}$$

Since $b, d, t \in \mathbb{N}$ we must have $bdt \in \mathbb{N}$. Since $a, b, c, d, s, t \in \mathbb{Z}$ we must have $adt + cbt + sbd \in \mathbb{Z}$. Thus the sum of the three rationals is again a rational.

Grading. Both points are for the solution. 1 point for the main algebraic step, and 1 point for the correct structure of this proof.

Q2.1 (2 POINTS)

Complete the following definition: "A natural number n is prime if ..."

Solution. $n \neq 1$ and the only divisors of p are 1 and p.

Grading. 1 point for the "not 1" qualifier (which may also be $n > 1$), and 1 point for the "divisor" part.

Q2.2 (3 POINTS)

Suppose that x is an integer with $x > 2$. Prove that $x^2 - 1$ is not a prime number.

Solution. Let $x \in \mathbb{Z}$ such that $x > 2$. Note that

$$x^2 - 1 = (x - 1)(x + 1),$$

with $x + 1, x - 1 \in \mathbb{Z}$ and both are divisors of $x^2 - 1$. Since $x > 2$, we must have that $x^2 > x$, so $x^2 - 1 > x - 1$.
Also $x > 2$ means $x - 1 > 1$. So $x - 1$ is an example of a divisor of $x^2 - 1$ that is not 1 or $x^2 - 1$.
So $x^2 - 1$ is not prime.

Grading. 1 point for the difference of squares. 1 point each for showing that $x - 1$ (or $x + 1$) is not 1 and not $x^2 - 1$. (They do not need to explicitly point out that $x - 1, x + 1$ are divisors of $x^2 - 1$.) For this question, *saying* that $x^2 - 1 \neq x - 1$ is enough; they don't need to prove it.

Instructions: You may use this page for any additional work you want graded. If you use this page, please indicate that on the original page containing the question.

[End of Quiz]