THE UNIVERSITY OF TORONTO - MISSISSAUGA - Quiz 2 - SOLNs

MAT102H5F - Fall 2019 - LEC0101-LEC0107

Time: 45 minutes

Date: Thursday October 10, 2019. 7:10PM - 7:55PM.

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Aids: None.

Instructions:

- Do not write on the QR code at the top of each page.
- Only the **front** of each page will be graded. (You may use the **backs** of pages for rough work.)
- You may use Page 5 for any additional work you want graded. If you use this page, please indicate that on the original page containing the question.
- Do not remove any pages.
- Your answers to the multiple choice questions must be recorded on the final page of the booklet.
- The quiz is out of 20 points.

Instructions: There are five (5) multiple choice questions worth two (2) points each. Choose the answer that is most correct. Your answers must be recorded on the final page of the booklet.

MC1 (2pts) Express the following statement by correctly using the logic symbols: "Every rational number x has a non-zero integer n such that xn is an integer."

A. $(\forall x \in \mathbb{Q})[xn = y](n \in \mathbb{Z} \setminus \{0\}, y \in \mathbb{Z})$

B. $(\forall x \in \mathbb{Q})(\forall n \in \mathbb{Z} \setminus \{0\})(\exists y \in \mathbb{Z})[xn = y]$

C. $(\forall x \in \mathbb{Q})(\exists n \in \mathbb{Z} \setminus \{0\})(\exists y \in \mathbb{Z})[xn = y] \longleftarrow$ [**Correct**.]

D. $(\forall x \in \mathbb{Q})(\exists n \in \mathbb{Z})(\exists y \in \mathbb{Z} \setminus \{0\})[xn = y]$

E. $(\forall x \in \mathbb{Q}) \wedge (\exists n \in \mathbb{Z} \setminus \{0\}) \wedge (\exists y \in \mathbb{Z} \setminus \{0\}) \wedge [xn = y]$

MC2 (2pts) Let $\emptyset$ be the empty set. Let $A = \{\emptyset\}$. Which statement is false?

A. $A \not\subseteq \emptyset$
B. $A \subseteq \emptyset \longleftarrow$ [**Correct**.]
C. $\emptyset \in A$
D. $\emptyset \subseteq A$
E. $\emptyset \neq A$

MC3 (2pts) Yixuan conjectures that: "If p is a prime, then p^2 is not a prime number". Which of the following proof strategies will prove this statement is true?

A. Note that $p = 3$ is prime, but $p^2 = 9$ is not prime.
B. Assume that p^2 is not prime, and conclude that p is prime.
C. Assume that p^2 is prime, and conclude that p is not prime. $\longleftarrow$ [**Correct**.]
D. Assume that p is prime, and conclude that p^2 is also prime.
E. Assume that p is not prime, so p^2 must also be not prime.

MC4 (2pts) Let $f : [-2, 2) \to \mathbb{R}$ be defined by $f(x) = x^2$. Which sets provide a counterexample to "$f(A \cap B) = f(A) \cap f(B)$"?

A. $A = \emptyset, B = \emptyset$.
B. $A = \{-1, 0\}, B = \{0, 1\} \longleftarrow$ [**Correct**.]
C. $A = [-2, 0], B = [0, 2]$.
D. $A = [-2, 2), B = [-2, 2)$.
E. $A = \mathbb{Q}, B = \mathbb{R} \setminus \mathbb{Q}$.

MC5 (2pts) Is $(P \Rightarrow P) \Rightarrow P$ logically equivalent to $P \Rightarrow (P \Rightarrow P)$?

A. Yes, they are both contradictions.
B. Yes, they are both tautologies.
C. Yes, but they are neither tautologies or contradictions.
D. No, but $(P \Rightarrow P) \Rightarrow P$ is a tautology.
E. No, but $P \Rightarrow (P \Rightarrow P)$ is a tautology. $\longleftarrow$ [**Correct**.]

Instructions: There are two (2) long answer questions worth five (5) points each, with multiple parts. Provide a complete solution, with justification, in the space provided.

Q1.1 (3 POINTS)

Write the negation of "$(\forall x, y \in \mathbb{R})(\exists z \in \mathbb{R})[(x < z) \Rightarrow (y < z)]$".

Solution. The following are logically equivalent, and the final statement is the desired negation.

$$\neg\Big((\forall x, y \in \mathbb{R})(\exists z \in \mathbb{R})[(x < z) \Rightarrow (y < z)]\Big)$$
$$(\exists x, y \in \mathbb{R})\neg\Big((\exists z \in \mathbb{R})[(x < z) \Rightarrow (y < z)]\Big)$$
$$(\exists x, y \in \mathbb{R})(\forall z \in \mathbb{R})\neg\Big([(x < z) \Rightarrow (y < z)]\Big)$$
$$(\exists x, y \in \mathbb{R})(\forall z \in \mathbb{R})[(x < z) \wedge (y \geq z)]$$

Grading.

(1) 1pt. Correctly negated all three quantifiers. (0/1 if any mistakes.)
(2) 1pt. Correctly negated $P \Rightarrow Q$ as $P \wedge \neg Q$.
(3) 1pt. Correctly negated $y < z$.

Q1.2 (2 POINTS)

Use proof by contradiction to prove that there are no rational solutions to

$$x^3 + x + 1 = 0.$$

Solution.

Proof by Contradiction: Assume that what has to be proven is false, i.e. The equation $x^3 + x + 1 = 0$ has rational solutions.

Assume that the equation has a rational solution $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$ such that a and b have no common factors.

Substituting $x = \dfrac{a}{b}$ in $x^3 + x + 1 = 0$ and then multiplying by b^3:

$$x^3 + x + 1 = 0$$
$$\left(\frac{a}{b}\right)^3 + \frac{a}{b} + 1 = 0$$
$$a^3 + ab^2 + b^3 = 0$$

We consider the following three cases:

(1) If a and b are odd, then a^3, ab^2, and b^3 are all odd. This implies that $a^3 + ab^2 + b^3$ is odd, and it cannot be 0.
(2) If a is odd and b is even, then a^3 is odd and ab^2 and b^3 are even. This implies that $a^3 + ab^2 + b^3$ is odd, and it cannot be 0.
(3) If a is even and b is odd, then a^3 and ab^2 are even and b^3 is odd. This implies that $a^3 + ab^2 + b^3$ is odd, and it cannot be 0.

Thus, a and b have to both be even integers.

This contradicts the assumption that a and b have no common factors. Hence, the equation $x^3 + x + 1 = 0$ has no rational solutions.

Grading.

(1) 1pt Getting to $a^3 + ab^2 + b^3 = 0$.
(2) 1pt Breaking into even/odd cases and successfully deriving contradictions.

Q2.1 (2 POINTS)

Let A, B be sets. Use only logic symbols $(\vee, \wedge, \Rightarrow, \Leftrightarrow, \neg, \in, \notin)$ to express the following statement: "$x \in (A \setminus B) \cup (B \setminus A)$ if ..."

Solution.

$$[x \in (A \setminus B)] \vee [x \in (B \setminus A)]$$
$$\Leftrightarrow [x \in A \wedge x \notin B] \vee [x \in B \wedge x \notin A]$$

Grading.

(1) 1pt. Turning the $x \in C \cup D$ into a $x \in C \vee x \in D$.
(2) 1pt. Turning both $x \in C \setminus D$ into $x \in C \wedge x \notin D$.

Q2.2 (3 POINTS)

Suppose that $A \subseteq B$ and $C \subseteq D$. Prove that $(A \cap C) \subseteq (B \cup D)$.

Solution.

Let x be in $A \cap C$, that is, $x \in (A \cap C)$. By definition of intersection, $x \in A$ and $x \in C$.

Given $x \in A \subseteq B$, $x \in A \Rightarrow x \in B$, by definition of inclusion of sets.

Given $x \in C \subseteq D$, $x \in C \Rightarrow x \in D$, by definition of inclusion of sets.

Combining the two statements, $x \in B$ and $x \in D$. Thus, $x \in (B \cap D)$, and $(B \cap D) \subseteq (B \cup D)$. It follows that $x \in (B \cup D)$.

Grading.

- 1pt. Correct structure of proof. (Assume $x \in A \cap C$, and show $x \in B \cap D$.)
- 1pt. Used the assumptions $A \subseteq B$ and $C \subseteq D$.
- 1pt. Combined $x \in B$ and $x \in D$ into $x \in B \cap D$.

Instructions: You may use this page for any additional work you want graded. If you use this page, please indicate that on the original page containing the question.

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