THE UNIVERSITY OF TORONTO - MISSISSAUGA - Quiz 3 - Version 1 - Solutions

MAT102H5F - Fall 2019 - LEC0101-LEC0107

Time: 45 minutes

Date: Thursday November 14, 2019. 7:10PM - 7:55PM.

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Aids: None.

Instructions: There are five (5) multiple choice questions worth two (2) points each. Choose the answer that is most correct. Your answers must be recorded on the final page of the booklet.

MC1 (2pts) Ali conjectures that "For every natural number $\sum_{i=1}^{n} i \leq \prod_{i=1}^{n} i$.".

Is his conjecture true?

A. Yes, because sums are always less than or equal to products.
B. Yes, because $i \leq i$ is always true.
C. No, because for $n = 4$ this is a strict inequality.
D. No, because the inequality is not true for $n = 1$.
E. No, because the inequality is not true for $n = 2$. $\leftarrow$ [Correct]

MC2 (2pts) Which of the following statements is a more general version of the observation that

$$1 + 4 + 7 + 10 + 13 = \frac{5(14)}{2}.$$

A. $\forall n \in \mathbb{N}, \sum_{i=1}^{n} (3i - 2) = \frac{n(3n-1)}{2}$. $\leftarrow$ [Correct]
B. $\forall n \in \mathbb{N}, \sum_{i=1}^{5} (3i - 2) = \frac{n(3n-1)}{2}$.
C. $\forall n \in \mathbb{N}, \sum_{i=1}^{n} (3i - 2) = \frac{i(3i-1)}{2}$.
D. $\sum_{i=1}^{13} (3i - 2) = \frac{5(14)}{2}$.
E. $\sum_{i=1}^{13} i = \frac{5(14)}{2}$.

MC3 (2pts) What is the range of the function $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ defined by $f(x) = \dfrac{|x| + x}{x^2}$?

A. $(0, \infty)$
B. $[0, \infty) \leftarrow$ [Correct]
C. $(-\infty, \infty)$
D. $[1, \infty)$
E. $\mathbb{R} \setminus \{0\}$

MC4 (2pts) Pranav has a mathematical statement $P(n)$. Pranav knows:

(1) $P(n)$ is true for all prime numbers n, and
(2) Whenever $P(n)$ is true, then $P(n-1)$ is also true.

The set all of numbers n that Pranav knows $P(n)$ is true is:

A. $\mathbb{N}$.
B. The set of all prime numbers.
C. $\mathbb{Z} \leftarrow$ [Correct]
D. The set of all natural numbers $n \geq 2$.
E. $\emptyset$.

MC5 (2pts) Which of the following is the induction hypothesis for strong induction when used to prove "$\forall n \in \mathbb{N}, P(n)$".

A. $P(1)$.
B. $P(n)$ for all $n \in \mathbb{N}$.
C. $P(n)$ for a particular $n \in \mathbb{N}$.
D. $P(1), \ldots, P(n)$ for all $n \in \mathbb{N}$.
E. $P(1), \ldots, P(n)$ for a particular $n \in \mathbb{N}$. $\leftarrow$ [Correct]

Instructions: There are two (2) long answer questions worth five (5) points each, with multiple parts. Provide a complete solution, with justification, in the space provided.

Q1.1 (3 POINTS)

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{2x}{1+x^2}$. Show that $\dfrac{1}{3}$ is an element of the range of f. (Do not use calculus.)

Solution. Note that

$$\begin{aligned} \frac{1}{3} = \frac{2x}{1+x^2} &\Leftrightarrow 1 + x^2 = 6x \\ &\Leftrightarrow x^2 - 6x + 1 = 0 \end{aligned}$$

By the quadratic formula, the solutions to this are

$$x = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm \sqrt{32}}{2} = 3 \pm \sqrt{8}.$$

So either of these x will give $f(x) = \frac{1}{3}$.

Grading. 1 point for using the quadratic formula. 1 point for giving an x that works (or justifying why one must exist). 1 point for the correct structure (do not award this point if they assume what they want to prove).
Assign 0/3 if the argument relies on calculus (continuity or IVT or something similar).

Q1.2 (2 POINTS)

Prove that

$$\sum_{i=1}^{4} i^2 = \sum_{i=1}^{4} i + \sum_{i=2}^{4} i + \sum_{i=3}^{4} i + \sum_{i=4}^{4} i.$$

Solution. [Brute force] Expanding both gives a sum of 30.

Solution. [Algebra] Note that

$$\begin{aligned}
\sum_{i=1}^{4} i + \sum_{i=2}^{4} i + \sum_{i=3}^{4} i + \sum_{i=4}^{4} i &= (1+2+3+4) \\
&+ (2+3+4) \\
&+ (3+4) \\
&+ (4) \qquad \text{Count the 1s, 2s, 3s, and 4s} \\
&= 1(1) + 2(2) + 3(3) + 4(4) \\
&= \sum_{i=1}^{4} i^2
\end{aligned}$$

Grading. 1 point for expanding the two sums correctly. 1 point for a correct argument.

Recall that the Fibonacci numbers are defined as

$$F_1 = 1, F_2 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for all } n \in \mathbb{N}.$$

Q2.1 (2 POINTS)

Is F_{303} an even number? Circle one of the following. (No proof needed)

$\boxed{\textbf{Yes}}$ **No**

Solution. No argument needed. The more general fact is that F_{3n} is even for all $n \in \mathbb{N}$. This is true because the Fibonacci numbers go: Odd, Odd, Even, Odd, Odd, Even, ... This can be proved formally by induction.

Grading. This is an all or nothing question. No argument is needed.

Q2.2 (3 POINTS)

Prove that for all $n \in \mathbb{N}$ that $F_1 + F_2 + \ldots + F_n = -1 + F_{n+2}$.

Solution. Let $P(n)$ be the statement "$F_1 + F_2 + \ldots + F_n = -1 + F_{n+2}$". We will prove this by induction.

$\boxed{\text{Base}}$ Note that $F_3 = F_2 + F_1 = 1 + 1 = 2$. Also:

$$-1 + F_3 = -1 + 2 = 1 = F_1.$$

So $P(1)$ is true.

$\boxed{\text{Inductive step}}$ Assume that $P(n)$ is true for a particular $n \in \mathbb{N}$. This means:

$$-1 + F_{n+2} = F_1 + F_2 + \ldots + F_n.$$

Note

$$\begin{aligned} -1 + F_{n+3} &= -1 + (F_{n+2} + F_{n+1}) && \text{by definition of } F_{n+3} \\ &= (-1 + F_{n+2}) + F_{n+1} \\ &= (F_1 + F_2 + \ldots + F_n) + F_{n+1} && \text{by IH} \\ &= F_1 + F_2 + \ldots + F_{n+1}. \end{aligned}$$

As desired. So $P(n+1)$ is true.

Grading.

- 0pt: Stated the $P(n)$ explicitly and correctly. [**This is not worth a point in this problem**.]
- 1pt: Proved the base case explicitly and correctly.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
- 1pt: The induction hypothesis was explicitly assumed for a particular $n \in \mathbb{N}$, and its use was pointed out (correctly).
- 1pt: The structure of the proof of the inductive step was correct.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
  - Award this point only if the mathematical idea of the inductive step is mostly correct.

Instructions: You may use this page for any additional work you want graded. If you use this page, please indicate that on the original page containing the question.

[End of Quiz]