THE UNIVERSITY OF TORONTO - MISSISSAUGA - Quiz 4 - Version 1 - Solutions
MAT102H5F - Fall 2019 - LEC0101-LEC0107

Time: 45 minutes

Date: Thursday November 21, 2019. 7:10PM - 7:55PM.

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Aids: None.

Instructions:

- Do not write on the QR code at the top of each page.
- You may use the **back** of the cover page, and pages 2-3 for rough work.
- You may use Page 6 for any additional work you want graded. If you use this page, please indicate that on the original page containing the question.
- Do not remove any pages.
- Your answers to the multiple choice questions must be recorded on the final page of the booklet.
- The quiz is out of 20 points.

Instructions: There are five (5) multiple choice questions worth two (2) points each. Choose the answer that is most correct. Your answers must be recorded on the final page of the booklet.

MC1 (2pts) Let $f : A \to B$ and $g : C \to D$ be functions. What condition is necessary for $g \circ f$ to be defined?

A. $A = D$,
B. $B = C$,
C. $B \subseteq C$,
D. range $g \subseteq A$,
E. range $f \subseteq C$. $\longleftarrow$ [**Correct**]

MC2 (2pts) Let M, N be natural numbers. What is:

$$\sum_{j=1}^{N} \left(\sum_{i=1}^{M} 1 \right)?$$

A. 1,
B. $M + N$,
C. $(M - 1)(N - 1)$,
D. MN, $\longleftarrow$ [**Correct**]
E. $(M + 1)(N + 1)$.

MC3 (2pts) How many functions $f : \{1, 2, \ldots, 2019\} \to \{0, 1\}$ are there that are <u>not</u> surjective?

A. 0,
B. 1,
C. 2, $\longleftarrow$ [**Correct**]
D. 2019,
E. 2^{2019}.

MC4 (2pts) The definition of a dyadic number is given recursively as follows:

(1) 0 is dyadic,

(2) 1 is dyadic, and

(3) Whenever a is dyadic and b is dyadic, then $\frac{a+b}{2}$ is dyadic.

Which of the following numbers is dyadic?

A. $\frac{1}{8}$, $\longleftarrow$ [**Correct**]

B. $\frac{1}{\sqrt{2}}$,

C. $\frac{1}{6}$,

D. $\frac{1}{3}$,

E. $\frac{1}{5}$.

MC5 (2pts) Let $f : \{-1, 0, 2\} \to \mathbb{R}$ be defined by $f(x) = x^2$. Is this function an injection?

A. No, because x^2 fails the vertical line test.

B. No, because $(1)^2 = (-1)^2$ and $-1 \neq 1$.

C. No, because $0^2 = 0$.

D. Yes, because x^2 passes the vertical line test.

E. Yes, because if $f(a) = f(b)$, then $a = b$. $\longleftarrow$ [**Correct**]

Instructions: There are two (2) long answer questions worth five (5) points each, with multiple parts. Provide a complete solution, with justification, in the space provided.

Recall that the Fibonacci numbers are defined by $F_1 = 1, F_2 = 2$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N}$.

Q1.1 (2 POINTS)

Prove (directly, without induction) that $F_{n+4} = 3F_{n+1} + 2F_n$, for every $n \in \mathbb{N}$.

Solution. Note

$$\begin{aligned} F_{n+4} &= F_{n+3} + F_{n+2} \\ &= (F_{n+2} + F_{n+1}) + (F_{n+1} + F_n) \\ &= F_{n+2} + 2F_{n+1} + F_n \\ &= (F_{n+1} + F_n) + 2F_{n+1} + F_n \\ &= 3F_{n+1} + 2F_n \end{aligned}$$

Grading. 1pt for correctly stating $F_{n+4} = F_{n+3} + F_{n+2}$. 1 pt for completing the argument correctly.
There is a more difficult solution that starts from $3F_{n+1} + 2F_n$ and gets F_{n+4}. This solution is also acceptable.

Q1.2 (3 points)

Prove by induction that for every natural number n, that F_{4n} is a multiple of 3. (You may use the result of Q1.1.)

Solution. We will prove this by induction. We will assume (the obvious fact) that every F_n is an integer.
Let $P(n)$ be the statement "F_{4n} is a multiple of 3".

Base case For $n = 1$, observe that $F_3 = 1 + 1 = 2$, and so $F_{4(1)} = F_4 = 2 + 1 = 3$, which is clearly a multiple of 3.

Induction step Suppose that $P(n)$ is true for a particular $n \in \mathbb{N}$. So there is an $m \in \mathbb{Z}$ such that $F_{4n} = 3m$.
Note

$$\begin{aligned} F_{4(n+1)} &= F_{4n+4} \\ &= 3F_{4n+1} + 2F_{4n} && \text{By Q1.1} \\ &= 3F_{4n+1} + 2(3m) && \text{By IH} \\ &= 3(F_{4n+1} + 2m) \end{aligned}$$

Since $F_{4n+1} + 2m \in \mathbb{Z}$, we have shown that $F_{4(n+1)}$ is a multiple of 3. So $P(n+1)$ is true.

Grading.

- 0pt: Stated the $P(n)$ explicitly and correctly. [**This is not worth a point in this problem**.]
- 1pt: Proved the base case explicitly and correctly.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
- 1pt: The induction hypothesis was explicitly assumed for a particular $n \in \mathbb{N}$, and its use was pointed out (correctly).
- 1pt: The structure of the proof of the inductive step was correct.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
  - Award this point only if the mathematical idea of the inductive step is mostly correct.

Q2.1 (2 points)

A function $f : \mathbb{R} \to \mathbb{R}$ is defined to be bounded above if

$$(\exists M \in \mathbb{R})(\forall x \in \mathbb{R})[f(x) \leq M].$$

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is bounded above, then f is not a surjection.

Solution. Suppose that f is bounded above. So there is an $M \in \mathbb{R}$ such that $f(x) \leq M$ for all real x. In particular, since $M+1$ is a real number, and $M+1 > M$, there is no real x such that $f(x) = M + 1$.
So f is not surjective.

Grading. 1 point for choosing $y = M + 1$, or another suitable real number. 1 point for arguing why that y is not the image of any x.
Do not award any points for variations of the imprecise argument "f is bounded above, so it can't reach anything above M." In order to receive any points, the solution must provide an example of a y that does not get achieved.

Q2.2 (3 points)

Let $f : A \to B$ be a bijection and $g : B \to A$ be a function such that $f \circ g \circ f(a) = f(a)$ for all $a \in A$. Prove that $g = f^{-1}$.

Solution. Since f is a bijection, f^{-1} exists, and is a bijection. Let $a \in A$. By applying f^{-1} to both sides of $f \circ g \circ f(a) = f(a)$, we get

$$g \circ f(a) = a.$$

We will show that $f^{-1}(b) = g(b)$ for all $b \in B$, which shows that $g = f^{-1}$. Let $b \in B$. Since f is a bijection (in particular a surjection), there is an $a \in A$ such that $f(a) = b$. By definition $f^{-1}(b) = a$.
By above, $g \circ f(a) = a$, and thus

$$g(b) = f^{-1}(b),$$

as desired.

Grading. 1 point for correctly showing $g \circ f(a) = a$. 1 point for a correct structure (This means that they must be attempting to show $f^{-1}(b) = g(b)$ for all $b \in B$). 1 point for a correct and clear argument.

[End of Quiz]