THE UNIVERSITY OF TORONTO - MISSISSAUGA
Term Test - Version 1 - Solutions
MAT102H5F - Fall 2019 - LEC0101-LEC0107
Time: 60 minutes
Date: Thursday October 31, 2019. 7:10PM - 8:10PM.
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Aids: None.

Instructions:

- Do not write on the QR code at the top of each page.
- Only the **front** of each page will be graded. (You may use the **backs** of pages for rough work.)
- You may use Page 7 for any additional work you want graded. If you use this page, please indicate that on the original page containing the question.
- Do not remove any pages.
- Your answers to the multiple choice questions must be recorded on the final page of the booklet.
- The test is out of 27 points.

Instructions: There are six (6) multiple choice questions worth two (2) points each. Choose the answer that is most correct. Your answers must be recorded on the final page of the booklet.

MC1 (2pts) Yuba conjectures that "If $ax^2 + bx + c$ is a polynomial where a, b, c are integers, then all of its roots are integers." This is:

A. True, because $(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2$.
B. True, because the discriminant will be an integer.
C. False, and $x^2 - 1$ is a counterexample.
D. False, and $x^2 - 2$ is a counterexample. $\longleftarrow$ [**Correct**.]
E. False, and $x^2 + \sqrt{2}x + 1$ is a counterexample.

MC2 (2pts) Is $P \wedge (Q \vee P)$ logically equivalent to P?

A. No, because P does not contain any "Q" statements.
B. No, for example when Q is true and P is false.
C. Yes, and they are both tautologies.
D. Yes, and they are both contradictions.
E. Yes, and they are neither tautologies or contradictions. $\longleftarrow$ [**Correct**.]

MC3 (2pts) Jana conjectures that "if $x \in \mathbb{Z}$ is even, then x is not odd". Which of the following is the proof by contradiction?

A. Assume that x is odd, and then conclude that x is not even.
B. Assume that x is odd, and then derive a contradiction.
C. Assume that x is even, and then derive a contradiction.
D. Assume that x is even, and that x is odd, and then derive a contradiction. $\longleftarrow$ [**Correct**.]
E. Assume that x is even, and that $x + 1$ is even, and then derive a contradiction.

MC4 (2pts) Negate the following statement:

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{Z})[(x > 0) \Rightarrow (y < x)]$$

A. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{Z})[(x > 0) \Rightarrow (y \geq x)]$
B. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{Z})[(x \leq 0) \Rightarrow (y < x)]$
C. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{Z})[(x \leq 0) \wedge (y < x)]$
D. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{Z})[(x \leq 0) \wedge (y \geq x)]$
E. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{Z})[(x > 0) \wedge (y \geq x)]$ $\longleftarrow$ [**Correct**.]

MC5 (2pts) Let $A = \mathbb{N} \times \mathbb{Z}$. Define

$$(a, b) \equiv (x, y) \text{ if and only if } a + b = x + y.$$

Is $\equiv$ an equivalence relation on A?
A. No, because $\mathbb{N} \neq \mathbb{Z}$.
B. No, because $\equiv$ is not reflexive.
C. No, because $\equiv$ is not symmetric.
D. No, because $\equiv$ is not transitive.
E. Yes, $\equiv$ is an equivalence relation. $\longleftarrow$ [**Correct**.]

MC6 (2pts) What is $\left(2^{2019} + 0^{2019} + 1^{2019} + 9^{2019}\right)$ congruent to modulo 10?
A. 0
B. 1
C. 2
D. 8 $\longleftarrow$ [**Correct**.]
E. 9

Instructions: There are three (3) long answer questions worth five (5) points each, with multiple parts. Provide a complete solution, with justification, in the space provided.

Q1.1 (3 POINTS)

Let A, B, C, D be sets. Prove that $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$.

Solution. Let $z \in (A \times C) \cup (B \times D)$. So $z \in A \times C$ or $z \in B \times D$.
So $z = (x, y)$ with $x \in A$ and $y \in C$ or $x \in B$ and $y \in D$.
So $x \in A$ or $x \in B$ and $y \in C$ or $y \in D$.
So $x \in A \cup B$ and $y \in C \cup D$.
So $z = (x, y) \in (A \cup B) \times (C \cup D)$.

Grading. 1 pt for the correct structure of the proof. Award this only if they start with an element of $(A \times C) \cup (B \times D)$ and end with an element of $(A \cup B) \times (C \cup D)$. 1pt for correct "definition unwinding". 1pt for the correct steps/argument.

Q1.2 (2 POINTS)

Moazzam conjectures that "For all sets A, B, C, D we have

$$(A \times C) \cup (B \times D) = (A \cup B) \times (C \cup D)."$$

Prove that their conjecture is true, or provide a counterexample (with computations).

Solution. This is false. For example, let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$.
So $(A \times C) \cup (B \times D) = \{(1, 3)\} \cup \{(2, 4)\} = \{(1, 3), (2, 4)\}$.
But, $(A \cup B) \times (C \cup D) = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$.
The sets are not equal because $(2, 3) \in (A \cup B) \times (C \cup D)$ and $(2, 3) \notin (A \times C) \cup (B \times D)$.

Grading. 1pt For giving a counterexample, and 1pt for showing that the sets are not equal.
Stating "this is false" without support is worth 0pts.

Q2.1 (2 POINTS)

State the Arithmetic Mean Geometric Mean inequality (AMGM) for real numbers a, b.

There are multiple equivalent, correct versions.

Solution. If $a, b \in \mathbb{R}$ and $ab \geq 0$, then $\frac{a+b}{2} \geq \sqrt{ab}$.

Solution. If $a, b \in \mathbb{R}$ and $\sqrt{ab}$ is defined, then $\frac{a+b}{2} \geq \sqrt{ab}$.

Solution. If $a, b \in \mathbb{R}$, then $\frac{(a+b)^2}{4} \geq ab$.

Grading. 1pt for the correct terms. If they have the correct terms, then award 1pt for the correct order of the inequality.
If they use the first or second solution, but fail to specify that the square root must be defined, then remove 0.5pts.

Q2.2 (3 POINTS)

Is $(2019)^2 \leq (2018)(2020)$ or is $(2018)(2020) \leq (2019)^2$? Make a conjecture and then prove your conjecture.

There are many acceptable solutions for this question.

Solution. Let $a = 2018, b = 2020$. Note that $\frac{a+b}{2} = 2019$. So by AMGM,

$$2019 = \frac{a+b}{2} \geq \sqrt{ab} = \sqrt{(2018)(2020)}.$$

Therefore, since all the terms are positive,

$$(2019)^2 \geq (2018)(2020).$$

Solution. Let $x = 2019$. Note that

$$(2018)(2020) = (x-1)(x+1) = x^2 - 1 \leq x^2 = (2019)^2.$$

Solution. Note $(2018)(2020) = 4076360 < 4076361 = (2019)^2$.

Grading. 1 pt for the correct inequality. 2pts for a correct argument.
Note, do not be generous for the third type of solution, since it is not a technique from this course.

Q3.1 (2 points)

Without using any "¬" symbols, write down the contrapositive of
"if $k \in \mathbb{Z}$ and $k \neq 0$, then $k \neq -k$".

Solution. The statement is of the form $(P \wedge R) \Rightarrow Q$.
So the contrapositive is $\neg Q \Rightarrow \neg(P \wedge R)$. This is equivalent to $\neg Q \Rightarrow \neg P \vee \neg R$.
This is:
If $k = -k$, then $k \notin \mathbb{Z}$ or $k = 0$.

Grading. Only the final statement is needed. 0.5pts for the $\neg Q \Rightarrow \neg(P \wedge R)$ part, and 0.5pts for each of the other three parts.

Q3.2 (3 points)

Let A be a set, and $B \subseteq A$. Suppose that $R = B \times B$ is an equivalence relation on A. Prove that $A = B$.

Solution. It is enough to show that $A \subseteq B$, since we are assuming $B \subseteq A$ in the statement of the question.
Let $x \in A$. Since R is an equivalence relation on A, then R is reflexive. Therefore $(x, x) \in R = B \times B$. By definition of the cross product, $x \in B$, as desired.

Grading. 1pt for starting with an element of A. 1 pt for observing that R is reflexive. 1pt for completing the argument.

Instructions: You may use this page for any additional work you want graded. If you use this page, please indicate that on the original page containing the question.

[End of Term Test]