MAT102H5 Y - SUMMER 2020 - PROBLEM SET 1

Comments

The following solutions were produced by me, Professor Mike Pawliuk. I do not claim that these are the *only* solutions. It's possible that there are correct solutions other than the ones I present. It's also possible to present the material in different ways.

The goal of this PS was to have students communicate clear solutions to challenging problems. Solutions should be

Grading. Each question is out of 5 points. In general we are looking for the following:

- Did the student solve the problem?
- Did the student communicate their solution clearly?

Problem 1. The syllabus for MAT102 says:

(P_1) Your "Best 5 (out of 6) problem sets" will make your problem set mark.
(P_2) "Your problem set mark will be determined by taking the average of the best five problem sets.".

(1) Compute P_1 and P_2 for the problem set grades: $10/15, 14/16, 1/10, 9/16$, [did not hand in PS5], $20/20$.
(2) Explain why (i.e. prove) no matter what 6 grades a student received on their problem sets, $P_1 = P_2$. (**Suggestion**: Your proof should start with "Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the grades a student has received on their (respective) problem sets.")

Problem Set	Grade [Out of 1]	Grade [out of 7]
1	$\frac{10}{15}$	$\frac{70}{15}$
2	$\frac{14}{16}$	$\frac{98}{16}$
3	$\frac{1}{10}$	$\frac{7}{10}$
4	$\frac{9}{16}$	$\frac{63}{16}$
5	Did not hand in	0
6	$\frac{20}{20}$	$\frac{140}{20}$
Average (without PS5):	≈ 0.6408	

Solution. [1.1] Each problem set is naturally given as a number between 0 and 1. To get the grade out of 7% we multiply by 7. In all cases we drop PS5, since it was not handed in, and is assigned a grade of 0.
Adding up column 3, we get $P1 \approx 22.43$ (out of a total possible 35).
To get P_2 we average the grades in Column 1 (without PS5) and then multiply by 35. So $P_2 \approx 22.43$.

Solution. [1.2] Let $x_1, x_2, \ldots, x_6$ be the grades the student has received on their respective problem sets. Assume that each number is given between 0 and 1. Let $y_1, y_2, \ldots, y_6$ be the grades ordered from highest to lowest. So we will drop y_6.
Notice (since each PS counts for 7%):

$$P1 = 7(y_1) + 7(y_2) + 7(y_3) + 7(y_4) + 7(y_5) = 7(y_1 + y_2 + y_3 + y_4 + y_5)$$

and, since the total assignment grade is 35%,

$$P2 = 35\frac{y_1 + y_2 + y_3 + y_4 + y_5}{5} = 7(y_1 + y_2 + y_3 + y_4 + y_5).$$

So $P1 = P2$.

Grading. Part 1 is worth 2 points:

- 1 for correct computation, and
- 1 for an answer given between 7 and 35.

Part 2 is worth 3 marks.

- 1 Point for the computation,
- 1 point for a clear explanation,
- 1 point for defining the x_i clearly.

Problem 2. A certain remote controlled car moves through the xy-plane by performing sequences of the following three moves:

- "$L(\theta)$", which is rotating θ degrees to the left. (Here $\theta \geq 0$ can be any real number.)
- "$R(\theta)$", which is rotating θ degrees to the right. (Here $\theta \geq 0$ can be any real number.)
- "$F(x)$", which is stepping forward x units in the direction it is facing. (Here $x \geq 0$ can be any real number.)

Assume the car starts at the origin $(0, 0)$ facing towards the positive x-axis. We want to know what points the car can reach. (In this question, "reach" means "reach and stop at", not "pass through".)

(1) Show that the car can reach the point $(-2, 2)$.
(2) Make a conjecture about what points the car can reach, and then prove your conjecture.
(3) Due to mild signal interference, the car can no longer accept instructions to make $R(\theta)$ moves (only $L(\theta)$ and $F(x)$ moves). Make a conjecture about what points the car can now reach, and then prove your conjecture.
(4) Due to severe signal interference, the car can only accept instructions to make $L(\theta)$ moves and $F(1)$ moves. Make a conjecture about what points the car can now reach, and then prove your conjecture.

The following is a solution to parts 1,2, and 3.

Theorem 1. Every point (a, b) can be reached by using a sequence of 4 moves of the types:

$$L(\theta_1), F(|a|), L(\theta_2), F(|b|).$$

Proof. Let (a, b) be any point in the plane. If $a, b \geq 0$, then the instructions are:

$$L(0), F(a), L(90°), F(b).$$

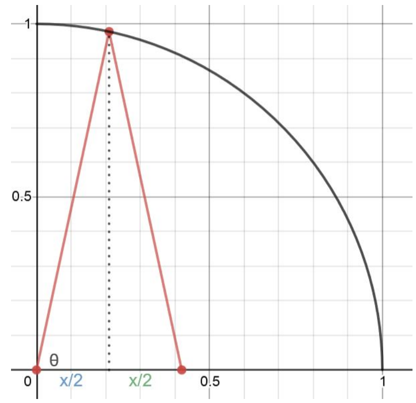
In other words, go straight a units, turn left, and go straight b units. The other instructions are similar:

- $a \geq 0, b < 0$. Instructions: $L(0), F(a), L(270°), F(|b|)$.
- $a, b < 0$. Instructions: $L(180°), F(|a|), L(90°), F(|b|)$.
- $a < 0, b \geq 0$. Instructions: $L(180°), F(|a|), L(270°), F(b)$.

□

In particular, to reach the point $(-2, 2)$ the instructions are: $L(180°), F(2), L(270°), F(2)$

Solution. [2.4] Note that it is possible to reach all points of the form (n, m) where $n, m \in \mathbb{Z}$ by using only moves of the type $L(90)$ and $F(1)$, as in the previous proof. All that remains is to show that we can reach all points (x, y) where $0 \leq x \leq 1$ and $0 \leq y \leq 1$. (Once you can reach those points in the unit square, you can reach any point (a, b) in the plane by first going to the point $(\lfloor a \rfloor, \lfloor b \rfloor)$, where $\lfloor a \rfloor$ is the largest integer less than a.)
Let $0 \leq x \leq 1$. The key idea is to use an isosceles triangle with side lengths 1, and base x, as in the diagram.



You can use the law of cosines to explicitly compute the angle θ if you wish. (In our question it is enough to argue that such an angle exists, we do not need to explicitly give it.) So the instructions will be $L(\theta), F(1), L(360 - 2\theta), F(1), L(\theta)$. The final $L(\theta)$ is to reset the car's orientation to its starting orientation (it is convenient, but not strictly necessary).
If you're interested, it's

$$x^2 = 1^2 + 1^2 - 1(1)(1)\cos\theta \Rightarrow \theta = \cos^{-1}\left(1 - \frac{x^2}{2}\right).$$

To reach any $0 \leq y \leq 1$, you use the same idea, but first rotate the car using a $L(90°)$ instruction.

Grading. This question is worth 5 points.

- 1 point each for each part. (Do not award part marks. Be generous)
- 1 point for clarity.

Problem 3. Question 1.5.31 (all parts) from the textbook. (You may assume, as the textbook says, that $\sqrt{7}, \pi, \log_2 3$ are irrational. If you claim that any other numbers are irrational, you should prove your claims.)

Solution.

Theorem 2. (1) If x is rational, then $-x$ is rational.
(2) If $-x$ is rational, then x is rational.
(3) x and $-x$ are either both rational, or both irrational.

Proof. 1. If $x = \frac{a}{b}$, where $a \in \mathbb{Z}, b \in \mathbb{N}$ then $-x = \frac{-a}{b}$, and $-a$ is an integer, since a is an integer. So $-x$ is rational.
2. The proof is similar.
Together we have that x is rational if and only if $-x$ is rational. So they are either both rational, or both irrational. □

Corollary 1. $-\sqrt{7}$ is irrational.

Fact 1. If n is an integer, then n is a rational number.

Proof. If n is an integer, then $n = \frac{n}{1}$, and 1 is a natural number. □

Corollary 2. 0 and 1 are rational numbers.

(1) Note that 1 is rational (Cor 2), but $\sqrt{1} = 1$ is rational (Cor 2).
(2) Note that $\sqrt{7}$ is irrational (by textbook assumption) and $-\sqrt{7}$ is irrational (by Cor 1), and $\sqrt{7}+(-\sqrt{7}) = 0$ which is rational (Cor 2).
(3) Note that 0 is an integer, and $\sqrt{7}$ is irrational (by textbook assumption), but $0(\sqrt{7}) = 0$ which is rational (by Cor 2).
(4) Note that $\sqrt{7}$ is irrational (by textbook assumption), and $\dfrac{\sqrt{7}}{\sqrt{7}} = 1$ is rational (by Cor 2).
(5) Let $a = \sqrt{7}$ which is irrational (by textbook assumption), and let $b = 1$ which is nonzero and rational (by Cor 2). Note that $\dfrac{a}{b} = \frac{\sqrt{7}}{1} = \sqrt{7}$ is irrational (by textbook assumption).

Grading. Each part is worth 1. Max 4/5 if they did not justify why $-\sqrt{7}$ is irrational (or whatever they used in part 2).

Problem 4. A tutorial group of MAT102 is playing the "MAT102 Warrior" game. The TA has prepared some "MAT102 warrior" tags, and has put the k tags on the back of t-shirts of k different students. The TA will then say: "MAT102 warrior, assemble!". If a student figures out that they have a tag on their back, they will go to the blackboard, otherwise they will not; If any of the "MAT102 Warriors" have not yet come to the blackboard, the TA will ask another round, and so on. When all students with "MAT102 warrior" tags come to the blackboard, the game is over.

We take the following as granted (*axioms*):

A1. The tutorial group has only finitely many students, but always more than the number of tags k;
A2. Every student knows the total number of tags k;
A3. k is a natural number;
A4. Every student can see the back of anyone except their own back;
A5. Everyone can see the blackboard;
A6. Every student in this tutorial group is identically extremely reasonable and clever.

We would like to study this game:

(1) Using the axioms, prove that each round after the TA summoned them, either (1) all the students with "MAT102 warriors" tag will figure out they are a "MAT102 warrior", or (2) none of them figures out they are a "MAT102 warrior". This will be your *lemma*.

(2) You make the following definitions:

Definition 1:
The **first assemble number** is the number of rounds it takes for the first "MAT102 warrior" to go to the blackboard.

Definition 2:
The **last assemble number** is the number of rounds it takes for the last "MAT102 warrior" to go to the blackboard.

Using the lemma you proved before, prove that **first assemble number**=**last assemble number**. We can henceforth call it the **assemble number**, which is well defined.

(3) If $k = 1$, what is the assemble number?
(4) If $k = 2$, what is the assemble number?
(5) If $k = 100$, what is the assemble number?

Now drop the axiom

"Every student knows the total number of tags k";

(6) If $k = 1$, what is the assemble number?
(7) If $k = 2$, what is the assemble number?
(8) If $k = 100$, make a conjecture on the assemble number? (you don't need to prove it now, as we will come back to the story of "MAT102 warriors" later in the semester.)

Solution.
The following theorem answers parts 2,3, and 4.

Theorem 3. Assume (A2). For $k \in \mathbb{N}$, if there are k tags, then the assemble number is 1.

Proof. Assume there are k tags. By (A2) all students know there are k tags. Each student counts the number of tags that they see (by A4, A6). They will each see either k tags (if they don't have a tag) or they will see $k-1$ tags (if they do have a tag).
When the TA says assemble for the first time, all those with tags will know they have tags (A6) and go to the blackboard. This proves that the last-assemble number and the first assemble number are both 1. □

Lemma 1. Assume (A2). The first-assemble number and the last-assemble number are equal. Moreover, they are both 1.

Proof. This is immediate from the previous theorem. □

Now assume we no longer know that (A2) is true.

Fact 2 (Non-A2.)**.** If a student sees k tags, then that student will <u>know</u> that there are either k or $k+1$ total tags.

Proof. This is because they can see all other students' backs (A4) and they either have a tag ($k+1$ total tags) or they don't (so k total tags). □

Fact 3 (Non-A2)**.** If a student sees 0 tags, they know they have a tag.

Proof. By Fact 2, this student knows that there are either 0 or 1 total tags. By (A3), k cannot be 0. So they conclude (A6) that they must have a tag. □

Theorem 4 (Non-A2)**.** For $k = 1$, the (non-A2) assemble number is 1.

Proof. Here the students will be in one of two cases:

- **Students without tags**. These students see 1 tag so they know there are either 1 or 2 total tags by Fact 2. During this round they will not act, since they are not sure if they have a tag or not.
- **Student with a tag**. This student sees 0 tags so by Fact 3 they know they have a tag. During this round they will go to the blackboard.

So in this case the assemble number is 1. □

Theorem 5 (Non-A2)**.** For $k = 2$, the (non-A2) assemble number is 2.

Proof. Here the students will be in one of two cases:

- **Students without tags**. These students see 2 tags so they know there are either 2 or 3 total tags by Fact 2. During this round they will not act, since they are not sure if they have a tag or not.
- **Students with tags**. These students see 1 tag so they know there are either 1 or 2 total tags by Fact 2. During this round they will not act, since they are not sure if they have a tag or not.

During round 1 no one goes to the blackboard. After round 1:

- **Students without tags**. These students see 2 tags so they know there are either 2 or 3 total tags by Fact 2. During this round they will not act, since they are not sure if they have a tag or not.
- **Students with tags**. These students see 1 tag so they know there are either 1 or 2 total tags by Fact 2. They deduce that there cannot be only 1 tag, because if there was only one tag, then the person with the tag would have gone to the blackboard in the previous round (Theorem 4). So they conclude that there are exactly 2 tags. Since they only see 1 other tag, they conclude that they have a tag. During this round they will go to the blackboard.

So the assemble number is 2. □

Conjecture. For $k \in \mathbb{N}$, the (non-A2) assemble number is k.

Grading. This question is worth 5 points total. Parts 2,3,4 + lemma (2 points), Parts 6,7 (1 point each). A correct conjecture in part 8 is worth 1 point.