MAT102H5 Y - SUMMER 2020 - PROBLEM SET 2

Submission

- **You must submit your completed problem set on Crowdmark by 5:00 pm (EDT) Friday May 29, 2020**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If you require additional space, please insert extra pages.
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- **You must include a signed and completed version of this cover page**.

Additional Instructions

You must justify and support your solution to each question.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

By signing this statement I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters.*

By signing this form I agree that the statements above are true.

If I do not agree with the statements above, I will not submit my assignment and will consult my course instructor immediately.

Student Name: ____

Student Number: ____

Signature: ____

Date: ____

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

TABLE 1. Table for Problem 1.2

Problem 1. (1) Show that $P \implies Q, P \Leftrightarrow Q$ and $P \vee Q$ are each logically equivalent to statements that only use (any number of) $P, Q, \wedge$, and $\neg$.

(2) There are 16 possible output columns for truth tables involving only P and Q. For example Table 1 shows that $P \wedge Q$ makes the column TFFF and $P \vee Q$ makes the column TTTF. For each of the 16 possible columns, find a logical statement with (any number of) $P, Q, \wedge, \vee, \implies, \Leftrightarrow, \neg$ that outputs that column. (For this question we expect you to list out 16 logical statements.)

(3) There are 256 possible output columns for truth tables involving only P, Q, and R. For each of the 256 possible columns, find a logical statement with (any number of) $P, Q, R, \wedge, \vee, \implies, \Leftrightarrow, \neg$ that outputs that column. (Hint: Try special cases and prove useful lemmas along the way. Aim for a solution that fits on one page.)

For this question, the grader will pick their favourite column and your solution should clearly and quickly tell them how to construct the logical statement that produces this column. The grader should also be convinced that the logical statement you provide will actually work.

Problem 2. Exercise 3.7.30.d from the textbook.

Problem 3. Consider the following function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = ax^6 + b\cos(x) + c|x| + d, \quad a, b, c, d \in \mathbb{R}$$

(1) Suppose that the equation $f(x) = 0$ has odd number of solutions. Conjecture the relation between b and d and formulate it into a mathematical statement of the type $P \Rightarrow Q$.
(2) What is the contrapositive of your implication in part 1?
(3) Prove your conjecture.

Problem 4. (1) **Pigeonhole Principle**
Suppose that m pigeons fly into n pigeonholes (boxes), with $m, n \in \mathbb{N}$ and $m > n$. Then prove by contradiction that at least one of these pigeonholes must have at least 2 pigeons in it.

(2) **Application**
A student named Jr. Proofini has 37 days to prepare for the final exam of MAT102. Instructors Mike and Qun give him some suggestions:
(a) In the next 37 days, he should study 60 hours in total;
(b) He should work at least 1 hour each day.
(c) Each day he should work several whole hours (i.e., he should work k_i hour(s) on the i^{th} day, $k_i \in \mathbb{N}$).
Jr. Proofini will follow this advice. We aim to prove the following statement, denoted as P:

Statement P

"No matter how Jr. Proofini arranges his working plan, there will be some interval of consecutive day(s) during which he worked for exactly 13 hours in total."

(a) Give an example of a possible studying plan that satisfies the requirements, and convince yourself the validity of P in your example.

Extra challenge, not for grades: If you are comfortable with programming (Python, Julia, R, Matlab e.t.c.), write code to simulate a random studying scheme of Jr. Proofini. Verify that P is true in these cases. Write code that verifies P is true for any random studying scheme.

(b) Can the observation you have achieved above, with/without the help of computer, be considered as a mathematical proof? Why?
(c) For all natural numbers $1 \leq i \leq 37$, let $a_i = k_1 + k_2 + \ldots + k_i$. Prove that $\forall 1 \leq i \leq 37$ we must have $1 \leq a_i \leq 60$.
(d) For all natural numbers $1 \leq i \leq 37$, let $b_i = a_i + 13$. Show that $14 \leq b_i \leq 73$.
(e) Now consider the numbers from 1 to 73 as pigeonholes, and the numbers $\{a_i\}_{1 \leq i \leq 37}$, $\{b_i\}_{1 \leq i \leq 37}$ as pigeons. Use the pigeonhole principle to conclude your proof.

Remark: As you can see, such proof, which is essentially based on proof by contradiction, gives a powerful result to general situation. However, it does not tell you the exact interval of day(s) Jr. Proofini has worked 13 hours. This is different compared to the constructive method in mathematical proof, which not only gives the existence of an object, but its explicit description as well.