MAT102H5 Y - SUMMER 2020 - PROBLEM SET 2 - SOLUTIONS

Comments

The following solutions were produced by the instructors Professor Mike Pawliuk and Dr Qun Wang. We do not claim that these are the *only* solutions. It's possible that there are correct solutions other than the ones we present. It's also possible to present the material in different ways.

Grading. Each question is out of 5 points. In general we are looking for the following:

- Did the student solve the problem?
- Did the student communicate their solution clearly?

Problem 1. (1) Show that $P \implies Q, P \Leftrightarrow Q$ and $P \vee Q$ are each logically equivalent to statements that only use (any number of) $P, Q, \wedge$, and $\neg$.

(2) There are 16 possible output columns for truth tables involving only P and Q. For each of the 16 possible columns, find a logical statement with (any number of) $P, Q, \wedge, \vee, \implies, \Leftrightarrow, \neg$ that outputs that column. (For this question we expect you to list out 16 logical statements.)

(3) There are 256 possible output columns for truth tables involving only P, Q, and R. For each of the 256 possible columns, find a logical statement with (any number of) $P, Q, R, \wedge, \vee, \implies, \Leftrightarrow, \neg$ that outputs that column. (Hint: Try special cases and prove useful lemmas along the way. Aim for a solution that fits on one page.)

For this question, the grader will pick their favourite column and your solution should clearly and quickly tell them how to construct the logical statement that produces this column. The grader should also be convinced that the logical statement you provide will actually work.

Solution. [1.1] Here we use $A \equiv B$ to mean A is logically equivalent to B. Note that

$$P \implies Q \equiv \neg\neg(P \implies Q) \equiv \neg(P \wedge \neg Q)$$

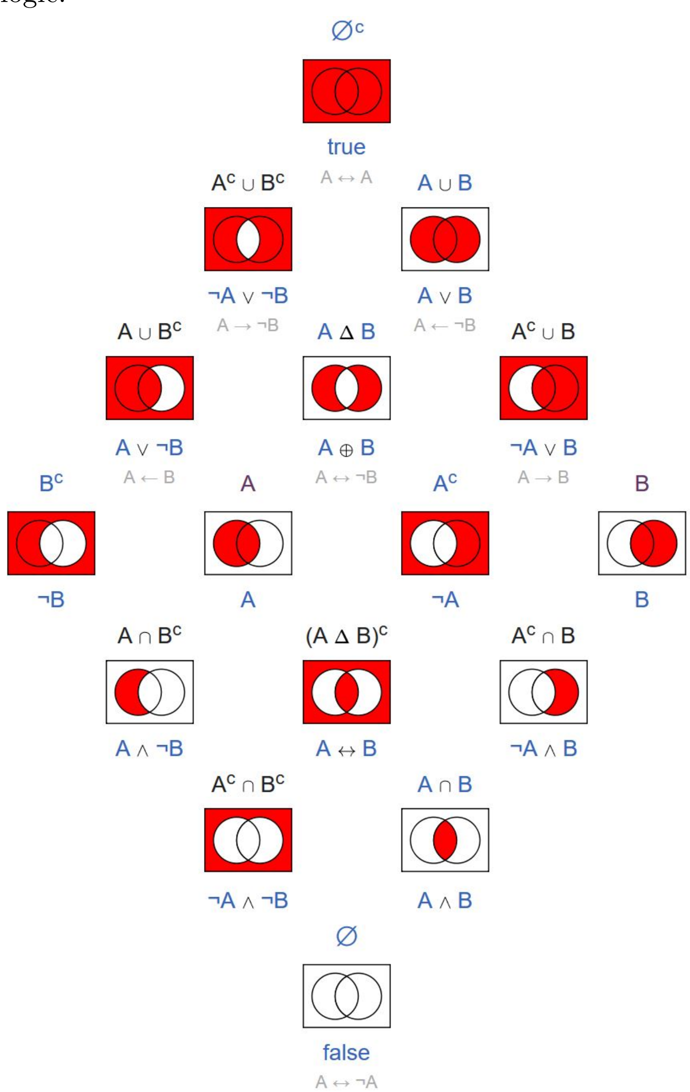
using this gives us

$$P \Leftrightarrow Q \equiv (P \implies Q) \wedge (Q \implies P) \equiv \neg(P \wedge \neg Q) \wedge \neg(Q \wedge \neg P)$$

and, by deMorgan's law

$$P \vee Q \equiv \neg\neg(P \vee Q) \equiv \neg(\neg P \wedge \neg Q)$$

Solution. [1.2] Here's a beautiful solution from Wikipedia. It shows off the connection between sets and logic.



`https://en.wikipedia.org/wiki/File:Venn0001.svg`

Solution. [1.3] The major idea here is that we will build up statements with any number of True outputs from 8 statements with only one True output. (Formally, we also remark that $P \wedge \neg P$ is a contradiction with no outputs of True.)

Lemma 1. The following 8 statements have exactly one output of True for each of the 8 possible assignments of True/False to P, Q, R:

(1) $TFFF\ FFFF : P \wedge Q \wedge R$

(2) $FTFF\ FFFF : P \wedge Q \wedge \neg R$

(3) $FFTF\ FFFF : P \wedge \neg Q \wedge R$

(4) $FFFT\ FFFF : P \wedge \neg Q \wedge \neg R$

(5) $FFFF\ TFFF : \neg P \wedge Q \wedge R$

(6) $FFFF\ FTFF : \neg P \wedge Q \wedge \neg R$

(7) $FFFF\ FFTF : \neg P \wedge \neg Q \wedge R$

(8) $FFFF\ FFFT : \neg P \wedge \neg Q \wedge \neg R$

Theorem 1. *For every sequence of truth value outputs (for three inputs P, Q, R) there is a statement with exactly that truth value output.*

Proof. Let $V = V_1V_2V_3V_4\ V_5V_6V_7V_8$ be a possible sequence of truth value outputs.
If V is all Falses, then use the statement $P \wedge \neg P$, which is a contradiction.
If V has at least one True, then "OR together" the appropriate statements from the lemma. That is, if V_i is True, use statement i in the lemma. For example, if V_1, V_3 and V_7 are True ($V = TFTF\ FFTF$), then we OR together the first, third and seventh statement from the lemma to get:

$$(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$$

which has the desired truth value output

$$TFTF\ FFTF.$$

Since OR evaluates to True if and only if at least one of the terms is True, and the statements in the lemma are true in exactly one spot (that aligns with V), V will be the precise truth value output of the statement we have created.

□

Solution. [1.3 Alternate] This solution is due to the TA Ivan Khatchatourian.
Let $V = V_1V_2V_3V_4\ V_5V_6V_7V_8$ be a possible sequence of truth value outputs. By Question 1.2, there is a statement X_1 (using only P, Q, no R) that has truth value output $V_1V_3V_5V_7$. By Question 1.2, there is a statement X_2 (using only P, Q, no R) that has truth value output $V_2V_4V_6V_8$.
Note that $(X_1 \wedge R) \vee (X_2 \wedge \neg R)$ will have exactly the desired output V. (If R is True, $X_1 \wedge R$ will produce the desired outputs for rows $1, 3, 5$, and 7, since the term $X_2 \wedge \neg R$ will be False. Similarly, if R is False, $X_2 \wedge \neg R$ will produce the desired outputs for rows $2, 4, 6$, and 8, since the term $X_1 \wedge R$ will be False.)
To illustrate this, suppose $V = TFTF\ FFTF$. Here, for the odd rows we have $TTFT$ and so use $X_1 = \neg P \vee Q$. For the even rows we have $FFFF$ and so use $X_2 = P \wedge \neg P$. Together this gives the statement:

$$((\neg P \vee Q) \wedge R) \vee (P \wedge \neg P \wedge \neg R).$$

Grading. This question is worth 5 points.

- Part 1 is 1 point (all or nothing).
- Part 2 is 2 points. Award 1 point if they gave math statements for all 16, and award another point if it seems mostly correct (You don't need to check that all 16 statements work.).
- Part 3 is worth 2 points: 1 point for a plausible idea, and another for a clear explanation.

Problem 2. Exercise 3.7.30.d from the textbook.

Solution. We prove this by contrapositive. Let $x, y \in \mathbb{R}$. Assume $\frac{x}{\sqrt{x^2+1}} = \frac{y}{\sqrt{y^2+1}}$.
Squaring both sides gives:

$$\frac{x^2}{x^2+1} = \frac{y^2}{y^2+1}.$$

Multiplying both sides by $(x^2+1)(y^2+1)$ (which is not zero as $x^2+1, y^2+1 \geq 1$) gives:

$$x^2(y^2+1) = y^2(x^2+1).$$

Expanding gives:

$$x^2y^2 + x^2 = y^2x^2 + y^2.$$

Cancelling gives:

$$x^2 = y^2.$$

This on its own only tells us that $x = y$ or $x = -y$. However, since $\frac{x}{\sqrt{x^2+1}} = \frac{y}{\sqrt{y^2+1}}$, and both denominators are positive, we must have that x and y have the same sign (or are both 0 and so $x = 0 = y$). Therefore, we know that $x \neq -y$ so we must have $x = y$, as desired.

Grading. This question is worth 5 points

- 2 points for the correct contrapositive.
- 1 point for the algebra.
- 1 point for a clear proof (connecting words, full sentences, etc.)
- 1 point for correctly excluding the $x = -y$ case.

Problem 3. Consider the following function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = ax^6 + b\cos(x) + c|x| + d, \quad a, b, c, d \in \mathbb{R}$$

(1) Suppose that the equation $f(x) = 0$ has odd number of solutions. Conjecture the relation between b and d and formulate it into a mathematical statement of the type $P \Rightarrow Q$.
(2) What is the contrapositive of your implication in part 1?
(3) Prove your conjecture.

Solution.

(1) Consider the following two statements
 P: $f(x) = 0$ has odd number of solutions
 Q: $b + d = 0$
 The conjecture is that $P \Rightarrow Q$, which reads:
 "If the equation $f(x) = 0$ has odd number of solutions, then $b + d = 0$"
(2) The contrapositive statement should be $\neg Q \Rightarrow \neg P$, where
 $\neg P$: $f(x) = 0$ has either infinitely many solutions, no solutions, or an even number of solutions.
 $\neg Q$: $b + d \neq 0$ The contrapositive statement reads:
 "If $b + d \neq 0$, then the equation $f(x) = 0$ has either infinitely many solutions, no solutions, or an even number of solutions."
(3) We prove the conjecture by using contrapositive.

 Proof. Suppose that $b + d \neq 0$. Note that $f(0) = b + d \neq 0$, which implies that 0 is not a solution. If the equation has no solution, or infinitely many solutions, we are done. Otherwise suppose the equation has finitely many solutions. Let $x = x_0$ be a solution, i.e. $f(x_0) = 0$, then one has that

$$0 = f(x_0) = ax_0^6 + b\cos(x_0) + c|x_0| + d = a(-x_0)^6 + b\cos(-x_0) + c|-x_0| + d$$

 It turns out that $x = -x_0$ is also a solution. Moreover $x_0 \neq 0$, which ensures that the solutions of $f(x)$ appear in pairs. As a result, there must be even number of solutions. □

Grading. This question is worth 5 points

- 1 point for the correct conjecture;
- 2 point for the correct contrapostive;
- 1 point for showing that $x = 0$ is not a solution from $b + d \neq 0$;
- 1 point for deduce from $b + d \neq 0$ that the solutions exist in pairs.

Problem 4. (1) **Pigeonhole Principle**

Suppose that m pigeons fly into n pigeonholes (boxes), with $m, n \in \mathbb{N}$ and $m > n$. Then prove by contradiction that at least one of these pigeonholes must have at least 2 pigeons in it.

(2) **Application**

A student named Jr. Proofini has 37 days to prepare for the final exam of MAT102. Instructors Mike and Qun give him some suggestions:

(a) In the next 37 days, he should study 60 hours in total;
(b) He should work at least 1 hour each day.
(c) Each day he should work several whole hours (i.e., he should work k_i hour(s) on the i^{th} day, $k_i \in \mathbb{N}$).

Jr. Proofini will follow this advice. We aim to prove the following statement, denoted as P:

Statement P

"No matter how Jr. Proofini arranges his working plan, there will be some interval of consecutive day(s) during which he worked for exactly 13 hours in total."

(a) Give an example of a possible studying plan that satisfies the requirements, and convince yourself the validity of P in your example.

Extra challenge, not for grades: If you are comfortable with programming (Python, Julia, R, Matlab e.t.c.), write code to simulate a random studying scheme of Jr. Proofini. Verify that P is true in these cases. Write code that verifies P is true for any random studying scheme.

(b) Can the observation you have achieved above, with/without the help of computer, be considered as a mathematical proof? Why?
(c) For all natural numbers $1 \leq i \leq 37$, let $a_i = k_1 + k_2 + \ldots + k_i$. Prove that $\forall 1 \leq i \leq 37$ we must have $1 \leq a_i \leq 60$.
(d) For all natural numbers $1 \leq i \leq 37$, let $b_i = a_i + 13$. Show that $14 \leq b_i \leq 73$.
(e) Now consider the numbers from 1 to 73 as pigeonholes, and the numbers $\{a_i\}_{1 \leq i \leq 37}$, $\{b_i\}_{1 \leq i \leq 37}$ as pigeons. Use the pigeonhole principle to conclude your proof.

Remark: As you can see, such proof, which is essentially based on proof by contradiction, gives a powerful result to general situation. However, it does not tell you the exact interval of day(s) Jr. Proofini has worked 13 hours. This is different compared to the constructive method in mathematical proof, which not only gives the existence of an object, but its explicit description as well.

Solution.

(1) We prove the pigeonhole principle by contradiction. Suppose to the contrary that no pigeonhole has more than 1 pigeon. Then the total number of pigeons should be at most n, which is strictly smaller than m. This is a contradiction. As a result there exists at least one pigeonhole which has at least two pigeons.

(2) Now we apply pieonhole principle to solve the problem.

(a) This question does not have unique solution. As long as the example constructed is reasonable it should be all right.

(b) It is not a mathematical proof unless the student has enumerated ALL possible studying plans (which is probably not the case...). The reason is that the statement actually carries a universal quantifier ("for any studying plan that meets the three conditions"), however enumerating some possible plans only provides the existential perspective. ("The statement is true for some plan")

(c) a_k is the total number of hours Proofini has worked in the first k days. As a result $1 \leq a_{k+1} - a_k, \forall 1 \leq k \leq 36$, which means that a_k is increasing as k becomes bigger. Now $a_1 \geq 1$, $a_{37} = 60$, the conclusion follows.

(d) Since $b_k = a_k + 13$, the conclusion follows from previous question.

(e) Consider $\{a_i\}_{1\leq i\leq 37}$, $\{b_i\}_{1\leq i\leq 37}$. These are 74 (not necessarily distinct) natural numbers. If we consider them as pigeons, the value for each of these numbers is between 1 and 73, hence in 73 pigeonholes. As a result some pigeonhole must has at least two pigeons. i.e. $b_{j_1} = a_{j_2}$ for some $j_1 < j_2$. (Note that you cannot have two numbers both from $\{a_i\}_{1\leq i\leq 37}$ or both from $\{b_i\}_{1\leq i\leq 37}$, as they are strictly increasing). But this means exactly that $a_{j_1} + 13 = a_{j_2}$. In other words

$$13 = a_{j_2} - a_{j_1} = (k_1 + k_2 + \ldots + k_{j_2}) - (k_1 + k_2 + \ldots + k_{j_1}) = k_{j_1+1} + k_{j_1+2} + \ldots + k_{j_2}$$

Which proves the result: Jr. Proofini has worked exactly 13 hours between day j_1 and day j_2.

Grading. This question is worth 5 points

- 1 point for the proof of pigeonhole principle;
- 1 point for construction of a reasonable working plan;
- 1 point for find the bound of $\{a_i\}_{1\leq i\leq 37}$ and that of $\{b_i\}_{1\leq i\leq 37}$
- 2 point for application of pigeonhole principle.