MAT102H5 Y - SUMMER 2020 - PROBLEM SET 3
SOLUTIONS

Submission

- **You must submit your completed problem set on Crowdmark by 5:00 pm (EDT) Friday June 12, 2020**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If you require additional space, please insert extra pages.
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- **You must include a signed and completed version of this cover page**.

Additional Instructions

You must justify and support your solution to each question.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

By signing this statement I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

By signing this form I agree that the statements above are true.
If I do not agree with the statements above, I will not submit my assignment and will consult my course instructor immediately.

Student Name: ____

Student Number: ____

Signature: ____

Date: ____

Advice. This is a serious problem set, and you will likely get stuck. That is intended; it is okay to get stuck. Getting unstuck is where the learning happens. We believe that you will be able to solve all questions in this problem set with determination, persistence, and play.
When you get stuck, please ask questions in Piazza, Office hours, before/after class or tutorial. We are here to help you succeed.

Problem 1. Given a set A, we define its power set, $\mathcal{P}(A)$, to be the set of all subsets of A: $\mathcal{P}(A) = \{B : B \subseteq A\}$. In this question you will discover the main idea in the proof of Claim 4.1.1 on page 88 of the textbook and express it in your own words. Please do not read p.87 or p.88 until after you have submitted your solutions; if/when you get stuck, instead please ask questions on Piazza or in office hours.

(1) Let $A = \{\text{Vincenzo, Yasmeen, Zipei}\}$. Find $\mathcal{P}(A)$;

Solution. Observe

$$\begin{aligned}\mathcal{P}(A) = \{&\emptyset, \{\text{ Vincenzo }\}, \{\text{ Yasmeen }\}, \{\text{ Zipei }\},\\ &\{\text{ Vincenzo, Yasmeen }\}, \{\text{ Vincenzo, Zipei }\}, \{\text{ Yasmeen, Zipei }\}, \{\text{ Vincenzo, Yasmeen, Zipei }\}\}\end{aligned}$$

(2) Let $B = \{\text{Vincenzo}, \text{Yasmeen}\}$. Find the set C such that $\mathcal{P}(B \cup \{\text{Zipei}\}) = \mathcal{P}(B) \cup C$, and C is disjoint from $\mathcal{P}(B)$.

Solution. Since $\mathcal{P}(B) = \{\emptyset, \{\text{ Vincenzo }\}, \{\text{ Yasmeen }\}, \{\text{ Vincenzo, Yasmeen }\}\}$, we see that

$$C = \{\{\text{ Zipei }\}, \{\text{ Vincenzo, Zipei }\}, \{\text{ Yasmeen, Zipei }\}, \{\text{ Vincenzo, Yasmeen, Zipei }\}\}.$$

(3) Continuing from part 2, show that C and $\mathcal{P}(B)$ have the same number of elements by pairing each element of $\mathcal{P}(B)$ with an element of C in a "natural" way.

Solution. For every set $X \in \mathcal{P}(B)$ the naturally paired set is $X \cup \{\text{Zipei}\} \in C$. This gives the following pairing:

$$\begin{aligned}\emptyset \quad & \{\text{ Zipei }\}\\ \{\text{ Vincenzo }\} \quad & \{\text{ Vincenzo, Zipei }\}\\ \{\text{ Yasmeen }\} \quad & \{\text{ Yasmeen, Zipei }\}\\ \{\text{ Vincenzo, Yasmeen }\} \quad & \{\text{ Vincenzo, Yasmeen, Zipei }\}\end{aligned}$$

Since every set in $\mathcal{P}(B)$ is different, and none contain Zipei, then all of the sets in C are different.

(4) Let A be the finite set $\{1, 2, ..., 2020\}$. Show that $\mathcal{P}(A \cup \{2021\})$ has twice as many elements as $\mathcal{P}(A)$.

Solution. Consider the pairing that associates each $X \in \mathcal{P}(A)$ to the set $X \cup \{2021\}$. Let $C = \{X \cup \{2021\} : X \in \mathcal{P}(A)\}$. Note that every element of $\mathcal{P}(A \cup \{2021\})$ either contains 2021 (and so is in C) or does not contain 2021 (and so is in $\mathcal{P}(A)$).
Since no $X \in \mathcal{P}(A)$ contains 2021, and all the elements of $\mathcal{P}(A)$ are different, we must have that all the elements of C are different. That means that there is exactly one element of C for each element of $\mathcal{P}(A)$. So $\mathcal{P}(A)$ and C have the same number of elements. So we have separated $\mathcal{P}(A \cup \{2021\})$ into two sets with an equal number of elements, and so it must have twice as many elements as $\mathcal{P}(A)$.

(5) Find, with proof, the number of elements of $\mathcal{P}(\{1, 2, \ldots, 2021\})$. Your answer should be a number. (Note: We are not expecting you to use induction for this question.)

Solution. Consider the sets $\mathcal{P}(\{1, 2, 3\}), \mathcal{P}(\{1, 2, 3, 4\}), \mathcal{P}(\{1, 2, 3, 4, 5\}), \ldots \mathcal{P}(\{1, 2, \ldots, 2021\})$. By Part 1.1, the first set has 8 elements, and then each subsequent set has double the number of elements. Since we double $2021 - 3 = 2018$ times, the final set will have $2^3 2^{2018} = 2^{2021}$ elements.

Grading. This question is worth 5 points total. 1 point for each question. Feel free to remove a point for lack of clarity.

Problem 2. Let $A \subseteq \mathbb{R}$.
A function $f : A \to A$ is called an embedding of A.
A function $f : A \to A$ is called a compression of A if $(\forall a \in A)[f(a) \leq a]$.
A function $f : A \to A$ is decreasing if $(\forall x \in A)(\forall y \in A)[x < y \implies f(x) \geq f(y)]$.

(1) There are 27 embeddings $f : \{1, 2, 3\} \to \{1, 2, 3\}$. Write them all down. (Hint: Find a way to represent each embedding using a three digit number.)

Solution. A function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is defined by specifying its values $f(1), f(2)$, and $f(3)$. We will represent this by creating a three digit number $f(1)f(2)f(3)$. For example, the function $f(1) = 2, f(2) = 3, f(3) = 1$ will be represented by 231.
Here is the complete list:

$$111, 112, 113, 121, 122, 123, 131, 132, 133,$$
$$211, 212, 213, 221, 222, 223, 231, 232, 233,$$
$$311, 312, 313, 321, 322, 323, 331, 332, 333$$

(2) Make a conjecture about the number of embeddings of the set $\{1, 2, 3, \ldots, 2020\}$ has. Prove your conjecture.

Solution. **Conjecture**. The set $\{1, 2, 3, \ldots, 2020\}$ has 2020^{2020} embeddings.
Proof. Every embedding f on this set is defined by its values $f(1), f(2), \ldots, f(2020)$. For each of these there are 2020 choices. Clearly any two such embeddings will be different. There are 2020^{2020} ways to make 2020 independent choices where each choice has 2020 options.

(3) The set $A = \{1, 2, 3\}$ has 6 compressions. Indicate them on your list from part 1.

Solution. Here they are:

$$\boxed{111}, \boxed{112}, \boxed{113}, \boxed{121}, \boxed{122}, \boxed{123}, 131, 132, 133,$$
$$211, 212, 213, 221, 222, 223, 231, 232, 233,$$
$$311, 312, 313, 321, 322, 323, 331, 332, 333$$

(4) Make a conjecture about the number of compressions the set $\{1, 2, 3, \ldots, 2020\}$ has. Prove your conjecture.

Solution. **Conjecture**. The set $\{1, 2, 3, \ldots, 2020\}$ has $2020!$ compressions, where $n! = (n)(n-1)\ldots(3)(2)(1)$.
Proof. Every compression f on this set is defined by its values $f(1), f(2), \ldots, f(2020)$. The compression condition imposes these restrictions:

- $f(1) \leq 1$, so it only has one choice.
- $f(2) \leq 2$, so it has two choices.
- $f(3) \leq 3$, so it has three choices.
- $\ldots$
- $f(2020) \leq 2020$, so it has 2020 choices.

Since all these choices are independent, this means that there are $(1)(2)(3)\ldots(2020) = 2020!$ different compressions.

(5) The set $A = \{1, 2, 3\}$ has 10 decreasing functions. Indicate them on your list from part 1.

Solution. Here they are:

$$\boxed{111}, 112, 113, 121, 122, 123, 131, 132, 133,$$
$$\boxed{211}, 212, 213, \boxed{221}, \boxed{222}, 223, 231, 232, 233,$$
$$\boxed{311}, 312, 313, \boxed{321}, \boxed{322}, 323, \boxed{331}, \boxed{332}, \boxed{333}$$

(6) Find a counterexample to the claim that every compression on $\{1, 2, 3\}$ is a decreasing function.

Solution. Using the notation defined in part 2.1, 112 is an example of a compression that is not decreasing.

(7) Find a counterexample to the claim that every decreasing function on $\{1, 2, 3\}$ is a compression.

Solution. Using the notation defined in part 2.1, 311 is an example of a decreasing function that is not a compression.

Grading. This question is worth 5 points total.

- 1 point for listing all the functions,
- 1 point total for listing all the compressions/decreasing functions. (You don't need to check everything closely; be generous)
- 1 point for a correct conjecture about number of embeddings, with any sort of reasonable argument. (Don't read the proof carefully.)
- 1 point for a correct conjecture about number of compressions, with any sort of reasonable argument. (Don't read the proof carefully.)
- 1 point together for the two counterexamples

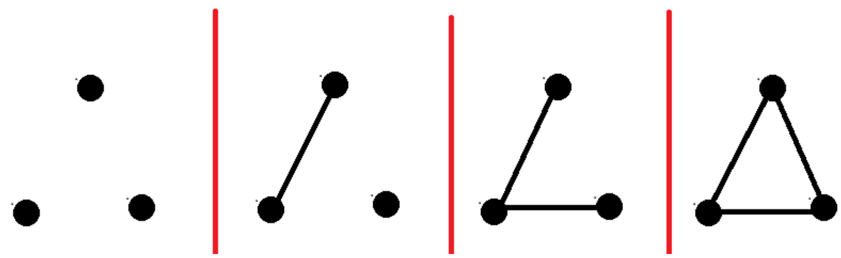
Problem 3. Let V be a set, and let E be a symmetric relation on V. We call any such pairing $G = (V, E)$ a graph, and call the elements of V the vertices, and the elements of E the edges.
By convention,

- We don't allow an edge to start and end at the same vertex (these would be called "loops").
- We visually represent an edge as a single curve connecting two vertices (instead of using two directed arrows).

We say that two graphs are "isomorphic" if you can get one from the other by relabeling the points. For example, on the vertex set $\{1, 2, 3\}$ the edge relations $E_1 = \{(1, 2), (2, 1), (3, 2), (2, 3)\}$ and $E_2 = \{(1, 3), (3, 1), (2, 3), (3, 2)\}$. In other words, two graphs are "the same" if they "look the same" when you ignore the labels on the vertices.

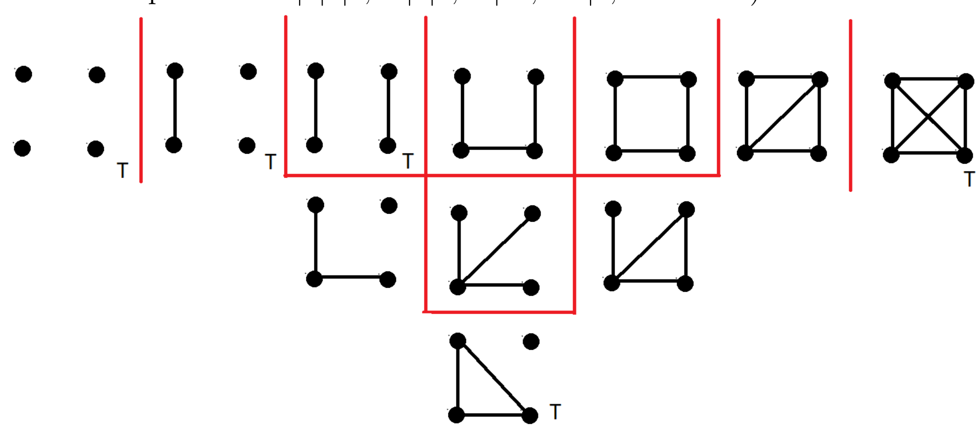
(1) There are 4 possible graphs on the vertices $\{1, 2, 3\}$. Find them. (You do not need to prove your result.)

Solution. Here they are:



(2) Find all possible graphs on the vertices $\{1, 2, 3, 4\}$. Identify all such graphs whose relation E is transitive after adding in all the loops. (You do not need to prove your result. Note: If you add in the loops you will have found all types of equivalence relations on $\{1, 2, 3, 4\}$.)

Solution. Here are all 11 of them. The ones that are transitive after adding loops are indicated with a "T". The columns represent the number of edges. (For fun: They correspond to the partitions $1|2|3|4$, $12|3|4$, $12|34$, $123|4$, and 1234.)



Grading. This question is worth 5 points.

- Part 1. 1 point for a partial list, and 1 point for a complete list.
- Part 2. 1 point for a partial list, and 1 point for a complete list. 1 Additional point for identifying all the transitive graphs.

Problem 4. In this problem we study how to construct new equivalence relations from known ones. This question is abstract, so it can be helpful to create many concrete examples for yourself to help understand the notation and concepts. For each question start by generating examples on a small set (e.g. $\{1, 2, 3\}$).
Notation: If $\sim$ is a relation on X, then $(x, y) \in \sim$ is another equivalent way of saying $x \sim y$. We use both to mean "x is related to y".

(1) Let A be a set with two equivalence relations $\overset{1}{\sim}$ and $\overset{2}{\sim}$. Now consider the following relations on A, $\overset{3}{\sim}$ and $\overset{4}{\sim}$ defined by:

$$a \overset{3}{\sim} b \text{ , if } a \overset{1}{\sim} b \text{ and } a \overset{2}{\sim} b$$

$$a \overset{4}{\sim} b \text{ , if } a \overset{1}{\sim} b \text{ or } a \overset{2}{\sim} b$$

Are $\overset{3}{\sim}$ and $\overset{4}{\sim}$ equivalence relations on A? Prove it or give an counterexample.

Solution. **Claim 1**. $\overset{4}{\sim}$ is not always a transitive relation (and therefore not an equivalence relation).
Let $A = \{1, 2, 3\}$ and let $\overset{1}{\sim} = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3)\}$, and let $\overset{2}{\sim} = \{(1, 3), (3, 1), (1, 1), (2, 2), (3, 3)\}$. Both are equivalence relations on A. Note that $2 \overset{4}{\sim} 1$, since $2 \overset{1}{\sim} 1$, and $1 \overset{4}{\sim} 3$, since $1 \overset{2}{\sim} 3$. However, $2 \overset{4}{\not\sim} 3$ since $2 \overset{1}{\not\sim} 3$ and $2 \overset{2}{\not\sim} 3$. Therefore $\overset{4}{\sim}$ is not transitive.
Claim 2. $\overset{3}{\sim}$ is a reflexive relation.
Let $a \in A$. Since $\overset{1}{\sim}$ is reflexive, we have $a \overset{1}{\sim} a$. Since $\overset{2}{\sim}$ is reflexive, we have $a \overset{2}{\sim} a$. Therefore, by definition $a \overset{3}{\sim} a$.
Claim 3. $\overset{3}{\sim}$ is a symmetric relation.
Let $a, b \in A$. Assume $a \overset{3}{\sim} b$. By definition, $a \overset{1}{\sim} b$ and $a \overset{2}{\sim} b$. Since $\overset{1}{\sim}$ is symmetric, we have $b \overset{1}{\sim} a$. Since $\overset{2}{\sim}$ is symmetric, we have $b \overset{2}{\sim} a$. Therefore, by definition, $b \overset{3}{\sim} a$.
Claim 4. $\overset{3}{\sim}$ is a transitive relation.
Let $a, b, c \in A$. Assume $a \overset{3}{\sim} b$ and $b \overset{3}{\sim} c$. By definition, $a \overset{1}{\sim} b, a \overset{2}{\sim} b, b \overset{1}{\sim} c, b \overset{2}{\sim} c$. Since $\overset{1}{\sim}$ is transitive, we have $a \overset{1}{\sim} c$. Since $\overset{2}{\sim}$ is symmetric, we have $a \overset{2}{\sim} c$. Therefore, by definition, $a \overset{3}{\sim} c$.
Together, we see that $\overset{3}{\sim}$ is an equivalence relation on A.

(2) Let A, B be two sets, and f be a function $f : A \to B$. Now given an equivalence relation $\overset{B}{\sim}$ on B, define the following relation on A by

$$a_1 \overset{A}{\sim} a_2 \text{ , if } f(a_1) \overset{B}{\sim} f(a_2)$$

Is $\overset{A}{\sim}$ an equivalence relation? Prove it or give an counterexample.

Solution. We verifify that the relation $\overset{A}{\sim}$ satisfies the reflexivity, the symmetry, and the transitivity.

(a) $\overset{A}{\sim}$ is reflexive: $\forall a \in A$, $f(a) \overset{B}{\sim} f(a)$ due to the reflexivity of $\overset{B}{\sim}$. Hence by definition $a \overset{A}{\sim} a$;
(b) $\overset{A}{\sim}$ is symmetric: Suppose that $a_1 \overset{A}{\sim} a_2$, then by definition $f(a_1) \overset{B}{\sim} f(a_2)$. But by the symmetry of $\overset{B}{\sim}$ this implies that $f(a_2) \overset{B}{\sim} f(a_1)$, which in turn shows that $a_2 \overset{A}{\sim} a_1$;
(c) $\overset{A}{\sim}$ is transitive: Suppose that $a_1 \overset{A}{\sim} a_2$ and $a_2 \overset{A}{\sim} a_3$, then by definition $f(a_1) \overset{B}{\sim} f(a_2)$ and $f(a_2) \overset{B}{\sim} f(a_3)$. The transitivity of $\overset{B}{\sim}$ thus implies that $f(a_3) \overset{B}{\sim} f(a_1)$, which by definition means that $a_3 \overset{A}{\sim} a_1$.

We conclude that $\overset{A}{\sim}$ is indeed an equivalence relation.

(3) Let A, B be two sets with the equivalence relation $\overset{A}{\sim}$ for elements in A and the equivalence relation $\overset{B}{\sim}$ for elements in B. Now consider the set $X = A \times B$ with the relation $\overset{\times}{\sim}$ defined by:

$$(a_1, b_1) \overset{\times}{\sim} (a_2, b_2) \text{ , if } a_1 \overset{A}{\sim} a_2 \text{ and } b_1 \overset{B}{\sim} b_2$$

Is $\overset{\times}{\sim}$ an equivalence relation on X? Prove it or give an counterexample.

Solution. We verify that the relation $\overset{\times}{\sim}$ satisfies the reflexivity, the symmetry, and the transitivity.

(a) $\overset{\times}{\sim}$ is reflexive: $\forall (a,b) \in A \times B$, $a \overset{\text{A}}{\sim} a$ and $b \overset{\text{B}}{\sim} b$ due to the reflexivity of $\overset{\text{A}}{\sim}$ and that of $\overset{\text{B}}{\sim}$. Hence by definition $(a,b) \overset{\times}{\sim} (a,b)$;

(b) $\overset{\times}{\sim}$ is symmetric: Suppose that $(a_1,b_1) \overset{\times}{\sim} (a_2,b_2)$, then by definition $a_1 \overset{\text{A}}{\sim} a_2$ and $b_1 \overset{\text{B}}{\sim} b_2$. Now due to the symmetry of $\overset{\text{A}}{\sim}$ and that of $\overset{\text{B}}{\sim}$, one has that $a_2 \overset{\text{A}}{\sim} a_1$ and $b_2 \overset{\text{B}}{\sim} b_1$. Hence by definition $(a_2,b_2) \overset{\times}{\sim} (a_1,b_1)$;

(c) $\overset{\times}{\sim}$ is transitive: Suppose that $(a_1,b_1) \overset{\times}{\sim} (a_2,b_2)$ and $(a_2,b_2) \overset{\times}{\sim} (a_3,b_3)$, then by definition $a_1 \overset{\text{A}}{\sim} a_2$ and $a_2 \overset{\text{A}}{\sim} a_3$, moreover $b_1 \overset{\text{B}}{\sim} b_2$ and $b_2 \overset{\text{A}}{\sim} b_3$. Now due to the transitivity of $\overset{\text{A}}{\sim}$ and that of $\overset{\text{B}}{\sim}$, one has that $a_3 \overset{\text{A}}{\sim} a_1$ and $b_3 \overset{\text{B}}{\sim} b_1$. Hence by definition $(a_3,b_3) \overset{\times}{\sim} (a_1,b_1)$;

We conclude that $\overset{\times}{\sim}$ is indeed an equivalence relation.

Grading. This question is worth 5 points.

- 1 point for proof of $\overset{3}{\sim}$;
- 1 point for counter-example of $\overset{4}{\sim}$;
- 2 point for proof of $\overset{\text{A}}{\sim}$;
- 1 point for proof of $\overset{\times}{\sim}$.