MAT102H5 Y - SUMMER 2020 - PROBLEM SET 3
UPDATED JUNE 8, 2020

Submission

- **You must submit your completed problem set on Crowdmark by 5:00 pm (EDT) Friday June 12, 2020**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If you require additional space, please insert extra pages.
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- **You must include a signed and completed version of this cover page**.

Additional Instructions

You must justify and support your solution to each question.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

By signing this statement I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

By signing this form I agree that the statements above are true.
If I do not agree with the statements above, I will not submit my assignment and will consult my course instructor immediately.

Student Name: ____

Student Number: ____

Signature: ____

Date: ____

Advice. This is a serious problem set, and you will likely get stuck. That is intended; it is okay to get stuck. Getting unstuck is where the learning happens. We believe that you will be able to solve all questions in this problem set with determination, persistence, and play.
When you get stuck, please ask questions in Piazza, Office hours, before/after class or tutorial. We are here to help you succeed.

Problem 1. Given a set A, we define its power set, $\mathcal{P}(A)$, to be the set of all subsets of A: $\mathcal{P}(A) = \{B : B \subseteq A\}$. In this question you will discover the main idea in the proof of Claim 4.1.1 on page 88 of the textbook and express it in your own words. Please do not read p.87 or p.88 until after you have submitted your solutions; if/when you get stuck, instead please ask questions on Piazza or in office hours.

(1) Let $A = \{\text{Vincenzo, Yasmeen, Zipei}\}$. Find $\mathcal{P}(A)$;
(2) Let $B = \{\text{Vincenzo}, \text{Yasmeen}\}$. Find the set C such that $\mathcal{P}(B \cup \{\text{Zipei}\}) = \mathcal{P}(B) \cup C$, and C is disjoint from $\mathcal{P}(B)$.
(3) Continuing from part 2, show that C and $\mathcal{P}(B)$ have the same number of elements by pairing each element of $\mathcal{P}(B)$ with an element of C in a "natural" way.
(4) Let A be the finite set $\{1, 2, ..., 2020\}$. Show that $\mathcal{P}(A \cup \{2021\})$ has twice as many elements as $\mathcal{P}(A)$.
(5) Find, with proof, the number of elements of $\mathcal{P}(\{1, 2, \ldots, 2021\})$. Your answer should be a number. (Note: We are not expecting you to use induction for this question.)

Problem 2. Let $A \subseteq \mathbb{R}$.
A function $f : A \to A$ is called an embedding of A.
A function $f : A \to A$ is called a compression of A if $(\forall a \in A)[f(a) \leq a]$.
A function $f : A \to A$ is decreasing if $(\forall x \in A)(\forall y \in A)[x < y \implies f(x) \geq f(y)]$.

(1) There are 27 embeddings $f : \{1, 2, 3\} \to \{1, 2, 3\}$. Write them all down. (Hint: Find a way to represent each embedding using a three digit number.)
(2) Make a conjecture about the number of embeddings of the set $\{1, 2, 3, \ldots, 2020\}$ has. Prove your conjecture.
(3) The set $A = \{1, 2, 3\}$ has 6 compressions. Indicate them on your list from part 1.
(4) Make a conjecture about the number of compressions the set $\{1, 2, 3, \ldots, 2020\}$ has. Prove your conjecture.
(5) The set $A = \{1, 2, 3\}$ has 10 decreasing functions. Indicate them on your list from part 1.
(6) Find a counterexample to the claim that every compression on $\{1, 2, 3\}$ is a decreasing function.
(7) Find a counterexample to the claim that every decreasing function on $\{1, 2, 3\}$ is a compression.

This part is not for credit, and you do not need to submit it. Reflect: What is the relationship between solving problems for small cases and solving problems for large cases? If we had not told you to do part (1) how would you have approached part (2)? If we had not told you to do part (3) how would you have approached part (4)?

Problem 3. Let V be a set, and let E be a symmetric relation on V. We call any such pairing $G = (V, E)$ a graph, and call the elements of V the vertices, and the elements of E the edges.
By convention,

- We don't allow an edge to start and end at the same vertex (these would be called "loops").
- We visually represent an edge as a single curve connecting two vertices (instead of using two directed arrows).

We say that two graphs are "isomorphic" if you can get one from the other by relabeling the points. For example, on the vertex set $\{1, 2, 3\}$ the edge relations $E_1 = \{(1, 2), (2, 1), (3, 2), (2, 3)\}$ and $E_2 = \{(1, 3), (3, 1), (2, 3), (3, 2)\}$. In other words, two graphs are "the same" if they "look the same" when you ignore the labels on the vertices.

(1) There are 4 possible graphs on the vertices $\{1, 2, 3\}$. Find them. (You do not need to prove your result.)
(2) Find all possible graphs on the vertices $\{1, 2, 3, 4\}$. Identify all such graphs whose relation E is transitive after adding in all the loops. (You do not need to prove your result. Note: If you add in the loops you will have found all types of equivalence relations on $\{1, 2, 3, 4\}$.)

Problem 4. In this problem we study how to construct new equivalence relations from known ones. This question is abstract, so it can be helpful to create many concrete examples for yourself to help understand the notation and concepts. For each question start by generating examples on a small set (e.g. $\{1, 2, 3\}$).

Notation: If $\sim$ is a relation on X, then $(x, y) \in\sim$ is another equivalent way of saying $x \sim y$. We use both to mean "x is related to y".

(1) Let A be a set with two equivalence relations $\overset{1}{\sim}$ and $\overset{2}{\sim}$. Now consider the following relations on A, $\overset{3}{\sim}$ and $\overset{4}{\sim}$ defined by:

$$a \overset{3}{\sim} b \text{ , if } a \overset{1}{\sim} b \text{ and } a \overset{2}{\sim} b$$
$$a \overset{4}{\sim} b \text{ , if } a \overset{1}{\sim} b \text{ or } a \overset{2}{\sim} b$$

Are $\overset{3}{\sim}$ and $\overset{4}{\sim}$ equivalence relations on A? Prove it or give an counterexample.

(2) Let A, B be two sets, and f be a function $f : A \to B$. Now given an equivalence relation $\overset{\mathrm{B}}{\sim}$ on B, define the following relation on A by

$$a_1 \overset{\mathrm{A}}{\sim} a_2 \text{ , if } f(a_1) \overset{\mathrm{B}}{\sim} f(a_2)$$

Is $\overset{\mathrm{A}}{\sim}$ an equivalence relation? Prove it or give an counterexample.

(3) Let A, B be two sets with the equivalence relation $\overset{\mathrm{A}}{\sim}$ for elements in A and the equivalence relation $\overset{\mathrm{B}}{\sim}$ for elements in B. Now consider the set $X = A \times B$ with the relation $\overset{\times}{\sim}$ defined by:

$$(a_1, b_1) \overset{\times}{\sim} (a_2, b_2) \text{ , if } a_1 \overset{\mathrm{A}}{\sim} a_2 \text{ and } b_1 \overset{\mathrm{B}}{\sim} b_2$$

Is $\overset{\times}{\sim}$ an equivalence relation on X? Prove it or give an counterexample.

This part is not for credit, and you do not need to submit it. After you complete this problem set, reflect on how far you've come since the beginning of the course. How would you have reacted to this problem set on the first day of class?