# MAT102H5 Y - SUMMER 2020 - PROBLEM SET 4
# UPDATED JUNE 16, 2020

## Submission

- **You must submit your completed problem set on Crowdmark by 5:00 pm (EDT) Friday July 10, 2020**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If you require additional space, please insert extra pages.
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- **You must include a signed and completed version of this cover page**.

## Additional Instructions

You must justify and support your solution to each question.

## Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

By signing this statement I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

By signing this form I agree that the statements above are true.
If I do not agree with the statements above, I will not submit my assignment and will consult my course instructor immediately.

Student Name: ______________________________________________

Student Number: ______________________________________________

Signature: ______________________________________________

Date: ______________________________________________

**Problem 1.** In this problem you will investigate tangent lines of circles without the use of calculus. Do not use any calculus in this problem.

**Definition (only for MAT102).** A line $L$ is tangent to a circle $C$ if $C$ and $L$ intersect at exactly one point.

(1) Show that if $x^2 + y^2 = 1$, then $x + y \leq 2$. Draw a picture of this situation using a circle and a line.

(2) Find all real values of $c$ that make the following implication true, and prove your result: if $x^2 + y^2 = 1$, then $y \leq -x + c$. (Hint: Draw a picture of both curves, isolate for $y$, and use the discriminant in the quadratic formula.)

(3) What is the smallest $c$ that works for part 2? Find the point of intersection of the circle $x^2 + y^2 = 1$ and the line $y = -x + c$, for this smallest $c$. Draw a picture of the relationship between this circle and this line.

(4) Use the methods above to find the $c > 0$ that makes $y = -2020x + c$ a tangent line to $x^2 + y^2 = 1$.

(5) Let $m > 0$ be a real number. What is the $c > 0$ that makes $y = -mx + c$ a tangent line to $x^2 + y^2 = 1$. (You do not need to prove your result.)

**Problem 2.** In this problem you will prove, and give an application of, the Cauchy–Schwarz Inequality.

(1) Prove that for any $a, b, c, d \in \mathbb{R}$

$$(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$$

When does the equality hold?

(2) Assume that $a_1, b_1, a_2, b_2, \ldots, a_{2020}, b_{2020}$ are all real numbers. Now consider the functions, where $1 \leq i \leq 2020$ and $i \in \mathbb{N}$:

$$f_i : \mathbb{R} \to \mathbb{R}$$

$$f_i(x) = (a_i x + b_i)^2, \quad a_i, b_i \in \mathbb{R}$$

Let $F(x) = f_1(x) + f_2(x) + .. + f_{2020}(x)$. What is the range of $F(x)$? (Hint: $F(x)$ is a type of curve that you know.)

(3) From the previous question, deduce that

$$(a_1 b_1 + a_2 b_2 + ... + a_{2020} b_{2020})^2 \leq (a_1^2 + a_2^2 + ... + a_{2020}^2)(b_1^2 + b_2^2 + ... + b_{2020}^2)$$

When does the equality hold?

(4) In the previous 3 parts, was there anything special about the number 2020? Write down an analogous inequality that uses $n$ terms instead of 2020 terms. (No proof is needed.)

(5) Consider the ellipse $A$ defined by

$$A = \{(x, y) \in \mathbb{R}^2, \frac{x^2}{9} + \frac{y^2}{16} = 1\}$$

(a) What is the maximum value of $f(x, y) = |x + y|$ where $(x, y) \in A$?
At which point(s) on $A$ is the maximum achieved?

(b) What is the minimum value of $f(x, y) = x^4 + y^4$ where $(x, y) \in A$?
At which point(s) on $A$ is the minimum achieved?