MAT102H5 Y - SUMMER 2020 - QUIZ 1

Submission

- **You must submit your completed Quiz on Crowdmark by 6:00pm (EDT) Tuesday May 19, 2020**.
- Late submissions will not be accepted.
- You should start uploading your quiz no later than 5:45pm.
- If you require additional space, please insert extra pages.
- You do not need to print out this quiz; you may submit clear pictures/scans of your work on lined paper, or screenshots/PDFs of your work.

Additional Instructions

You must justify and support your solution to each question.

Permitted Resources

During the quiz:

(1) You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
(2) You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, completing problem sets).
(3) Do not use personal notes related to other material (e.g. notes created by studying external websites)
(4) Do not communicate with anyone other than the instructors.
(5) Do not use Piazza.
(6) Do not use any online resources other than Quercus and Crowdmark.

Academic Integrity

By submitting this quiz you affirm that your submission represents entirely your own efforts. You confirm that:

- You have not copied any portion of this work.
- You have not allowed someone else in the course to copy this work.
- You understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

Problem 1 [10 points]

We study the possibility of constructing an order for the set of rational numbers $\mathbb{Q}$. Recall that a number r is rational, if there exists $p \in \mathbb{Z}, q \in \mathbb{N}$ such that $r = \frac{p}{q}$. In the following, we only consider the case where r is a positive rational, i.e., $r > 0$. As a result, one can just use $p, q \in \mathbb{N}$ in the representation.

(1) Find another representation of $r = \frac{4}{5}$ (no proof needed).

(2) Prove that the difference of any two rational numbers is a rational number.

(3) Recall that $r = \frac{p}{q}$ is called an irreducible fraction of the rational number r, if the only natural number that divides both p and q is 1. We know that every positive rational number r can be uniquely written as an irreducible fraction $r = \frac{p}{q}$, with $p, q \in \mathbb{N}$. Now consider the following definition:

Definition 1. For a positive rational number $r = \frac{p}{q}$ written as an irreducible fraction, we call the integer $p + q$ the height of r, denoted by $h(r) = p + q$.

(a) Find all the positive rationals r with $h(r) < 7$. (No proof needed.)

(b) Prove that $\forall r$, $h(r) \geq 2$. What is the only value of r where $h(r) = 2$?

(c) Prove or give a counterexample to the following statement:

"For any positive rational numbers r_1 and r_2, if $r_1 > r_2$, then $h(r_1) > h(r_2)$."

Problem 2 [5 points]

Recall that we have seen that a mathematical statement is a sentence that is either true or false.

(1) State the definition of a tautology and a contradiction.

(2) Let P and Q be mathematical statements. Prove that

$$((\neg P) \implies Q) \wedge ((\neg P) \implies (\neg Q)) \wedge \neg P$$

is a contradiction.

Problem 3 [5 points]

Note: This uses the same notation as Q2 from PS1. Differences are underlined.

A certain remote controlled car moves through the xy-plane by performing sequences of the following three moves:

- "$L(\theta)$", which is rotating θ degrees to the left. (Here $\theta \geq 0$ can be any real number.)
- "$R(\theta)$", which is rotating θ degrees to the right. (Here $\theta \geq 0$ can be any real number.)
- "$F(1)$", which is stepping forward 1 unit in the direction it is facing.

Assume the car starts at the origin $(0, 0)$ facing towards the positive x-axis. Moreover the battery only permits the car to step forward at most twice. We want to know what points the car can reach. (In this question, "reach" means "reach and stop at", not "pass through".).

Let B be the circle which is centred at origin with radius 2, so you may assume that the car will not move out of circle B.

(1) Prove, by giving an explicit list of orders, that the car can reach the point $(1, 1)$.

(2) Make a conjecture about what points the car can reach, and then prove your conjecture.

(3) Due to mild signal interference, the car can no longer accept instructions to make $R(\theta)$ moves (only $L(\theta)$ and F(1) moves). Make a conjecture about what points the car can now reach (without a proof).