MAT102H5 Y - SUMMER 2020 - QUIZ 1 - SOLUTIONS

Submission

- **You must submit your completed Quiz on Crowdmark by 6:00pm (EDT) Tuesday May 19, 2020**.
- Late submissions will not be accepted.
- You should start uploading your quiz no later than 5:45pm.
- If you require additional space, please insert extra pages.
- You do not need to print out this quiz; you may submit clear pictures/scans of your work on lined paper, or screenshots/PDFs of your work.

Additional Instructions

You must justify and support your solution to each question.

Permitted Resources

During the quiz:

(1) You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
(2) You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, completing problem sets).
(3) Do not use personal notes related to other material (e.g. notes created by studying external websites)
(4) Do not communicate with anyone other than the instructors.
(5) Do not use Piazza.
(6) Do not use any online resources other than Quercus and Crowdmark.

Academic Integrity

By submitting this quiz you affirm that your submission represents entirely your own efforts. You confirm that:

- You have not copied any portion of this work.
- You have not allowed someone else in the course to copy this work.
- You understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

Problem 1 [10 points]

We study the possibility of constructing an order for the set of rational numbers $\mathbb{Q}$. Recall that a number r is rational, if there exists $p \in \mathbb{Z}, q \in \mathbb{N}$ such that $r = \frac{p}{q}$. In the following, we only consider the case where r is a positive rational, i.e., $r > 0$. As a result, one can just use $p, q \in \mathbb{N}$ in the representation.

(1) Find another representation of $r = \frac{4}{5}$ (no proof needed).

Solution. There are many answers to this. The simplest is $r = \dfrac{8}{10}$. We will also accept $r = \dfrac{-4}{-5}$.

Grading. This is 1 point; all or nothing.

(2) Prove that the difference of any two rational numbers is a rational number.

Solution. Let x, y be rational numbers. By definition, there are $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$ so that $x = \frac{a}{b}$ and $y = \frac{c}{d}$.
Now

$$x - y = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

Since a, b, c, d are all integers, so is $ad - bc$. Since b, d are both (positive) natural numbers, so is bd. Therefore $x - y$ is a rational number.

Grading. This is worth 2 points: 1 for the rough idea, and 1 for the clarity of the proof (defining all terms used, checking that bd is a natural number, etc.)
Note that the proof for sums is in the LEC0101 slides, so don't be surprised if many solutions look similar.

(3) Recall that $r = \frac{p}{q}$ is called an irreducible fraction of the rational number r, if the only natural number that divides both p and q is 1. We know that every positive rational number r can be uniquely written as an irreducible fraction $r = \frac{p}{q}$, with $p, q \in \mathbb{N}$. Now consider the following definition:

Definition 1. For a positive rational number $r = \frac{p}{q}$ written as an irreducible fraction, we call the integer $p+q$ the height of r, denoted by $h(r) = p + q$.

(a) Find all the positive rationals r with $h(r) < 7$. (No proof needed.)

Solution. By definition, for $r = \frac{p}{q}$, h(r) = p+q. As a result $h(r) < 7$ if and only if $p + q < 7$. One can enumerate all of them

(i) $p = 1$, q can be $1, 2, 3, 4, 5$, which gives $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$;
(ii) $p = 2$, q can be $1, 3$, which gives $2, \frac{2}{3}$;
(iii) $p = 3$, q can be $1, 2$, which gives $3, \frac{3}{2}$;
(iv) $p = 4$, q can be 1, which gives 4;
(v) $p = 5$, q can be 1, which gives 5.

Hence all rational numbers with height smaller than 7 are $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 2, \frac{2}{3}, 3, \frac{3}{2}, 4, 5\}$

Grading. This is worth 2 points. Correct interpretation of idea 1 point, complete enumeration 1 point.

(b) Prove that $\forall r$, $h(r) \geq 2$. What is the only value of r where $h(r) = 2$?

Solution. Let $r = \frac{p}{q}$ be the irreducible fraction, then $h(r) = p + q \geq 1 + 1 = 2$. The equality holds iff $p = q = 1$,i.e., $r = \frac{1}{1} = 1$.

Grading. This is worth 3 points. 2 points for the proof for $h(r) \geq 2$, 1 point for the condition of equality.

(c) Prove or give a counterexample to the following statement:

"For any positive rational numbers r_1 and r_2, if $r_1 > r_2$, then $h(r_1) > h(r_2)$."

Solution. This statement is not correct. We see that $\frac{1}{5} < \frac{3}{2}$ but $h(\frac{1}{5}) = 6 > h(\frac{3}{2}) = 5$, which is a counterexample.

Grading. This is worth 2 points. 1 point for claiming the statement is false, 1 point for giving a counterexample (Any other counterexample should count too).

Problem 2 [5 points]

Recall that we have seen that a mathematical statement is a sentence that is either true or false.

(1) State the definition of a tautology and a contradiction.

Solution. A **tautology** is a mathematical statement that is always true, a **contradiction** is a mathematical statement that is always false.

Grading. This is worth 2 points. 1 point for each definition.

(2) Let P and Q be mathematical statements. Prove that

$$((\neg P) \implies Q) \wedge ((\neg P) \implies (\neg Q)) \wedge \neg P$$

is a contradiction.

Solution. Let $R = ((\neg P) \implies Q) \wedge ((\neg P) \implies (\neg Q)) \wedge \neg P$. The truth table is constructed as the following:

P	Q	$\neg P$	$\neg Q$	$((\neg P) \implies Q)$	$(\neg P) \implies (\neg Q)$	R
T	T	F	F	T	T	F
T	F	F	T	T	T	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

As a result one sees that the statement R is always false. In other words, it is a contradiction.

Grading. This is worth 3 points. 1 point for decomposition of the statement, 1 point for the construction of truth table, 1 point for the conclusion (Or equivalently for claiming correctly what should be shown/proved) Alternatively if a student proceeds by explanation without the truth table, then he/she should still obtain the points given their reasoning is correct. However, in the latter case if the student has never mentioned explicitly an argument like "If P is false then $P \Rightarrow Q$ is always correct" in an "if... then..." connective, then he/she will lose 1 point.

Problem 3 [5 points]

Note: This uses the same notation as Q2 from PS1. Differences are underlined.

A certain remote controlled car moves through the xy-plane by performing sequences of the following three moves:

- "$L(\theta)$", which is rotating θ degrees to the left. (Here $\theta \geq 0$ can be any real number.)
- "$R(\theta)$", which is rotating θ degrees to the right. (Here $\theta \geq 0$ can be any real number.)
- "$F(1)$", which is stepping forward 1 unit in the direction it is facing.

Assume the car starts at the origin $(0, 0)$ facing towards the positive x-axis. Moreover the battery only permits the car to step forward at most twice. We want to know what points the car can reach. (In this question, "reach" means "reach and stop at", not "pass through".).

Let B be the circle which is centred at origin with radius 2, so you may assume that the car will not move out of circle B.

(1) Prove, by giving an explicit list of orders, that the car can reach the point $(1, 1)$.

Solution. Following the order $F(1) \to L(90°) \to F(1)$ the car will reach the point $(1, 1)$

Grading. This is worth 1 point. All or nothing. It is all right if a student used $\frac{\pi}{2}$ instead of 90°.

(2) Make a conjecture about what points the car can reach, and then prove your conjecture.

Solution. Conjecture: The car can reach any point inside or on the circle B.
Proof: Let B_1 be the circle centred at origin with radius 1. For any point C inside or on the circle B, draw a circle B_2, which is centred at C with radius 1. B_1 and B_2 will intersect at least at one point D. Then the trajectory $O \to D \to C$ is the desired path.

Grading. This is worth 3 points. 1 point for the conjecture and 2 points for the proof. Any other methods will also be valid, as long as the proof is reasonable and complete.

(3) Due to mild signal interference, the car can no longer accept instructions to make $R(\theta)$ moves (only $L(\theta)$ and F(1) moves). Make a conjecture about what points the car can now reach (without a proof).

Solution. Conjecture: The car can reach any point inside or on the circle B.

Grading. This is worth 1 point. All or nothing.