MAT102H5 Y - SUMMER 2020 - QUIZ 2 - SOLUTIONS

Submission

- **You must submit your completed Quiz on Crowdmark by 6:00pm (EDT) Tuesday June 2, 2020**.
- Late submissions will not be accepted.
- You should start uploading your quiz no later than 5:45pm.
- If you require additional space, please insert extra pages.
- You do not need to print out this quiz; you may submit clear pictures/scans of your work on lined paper, or screenshots/PDFs of your work.

Additional Instructions

You must justify and support your solution to each question.

Permitted Resources

During the quiz:

(1) You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
(2) You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, completing problem sets).
(3) Do not use personal notes related to other material (e.g. notes created by studying external websites)
(4) Do not communicate with anyone other than the instructors.
(5) Do not use Piazza.
(6) Do not use any online resources other than Quercus and Crowdmark.

Academic Integrity

By submitting this quiz you affirm that your submission represents entirely your own efforts. You confirm that:

- You have not copied any portion of this work.
- You have not allowed someone else in the course to copy this work.
- You understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

Problem 1 [5 points]

Consider the statement: "Let $x, y \in \mathbb{Z}$. If xy is even, then x is even or y is even."

(1) Write down the converse of the implication.

Solution. The converse of the statement reads:

"Let $x, y \in \mathbb{Z}$. If x is even or y is even, then xy is even. "

Grading. This part is worth 1 point. All or nothing.

(2) Write down the contrapositive of the implication.

Solution. The contrapositive of the statement reads:

"Let $x, y \in \mathbb{Z}$. If x is odd **and** y is odd, then xy is odd, "

Grading. This part is worth 1 point. All or nothing.

(3) Prove the statement by contrapositve.

Solution. Suppose that $x, y \in \mathbb{Z}$, with x odd and y odd. By definition, there exists $n_1, n_2 \in \mathbb{Z}$ s.t.

$$x = 2n_1 + 1, \quad y = 2n_2 + 1 \tag{1}$$

It turns out that

$$xy = (2n_1 + 1)(2n_2 + 1) = 2(2n_1n_2 + n_1 + n_2) + 1 = 2M + 1 \tag{2}$$

where $M = 2n_1n_2 + n_1 + n_2 \in \mathbb{Z}$. Thus by definition xy is an odd number.

Grading. This part is worth 3 point. 2 for practicing the definition of odd/even number, 1 for understanding the idea of proof by contrapositive.

Problem 2 [10 points]

(1) Let A, B be sets. Prove that $A \cup B = A \cup (B \setminus A)$. (You can use any logical identities without proof, but you must say which identities you use and where exactly you use them.)

Solution.

Proof. We prove that $A \cup B \subset A \cup (B \setminus A)$ and $A \cup (B \setminus A) \subset A \cup B$

(a) Proof of $A \cup B \subset A \cup (B \setminus A)$: $\forall x \in A \cup B$, $x \in A$ or $x \in B$. Suppose that $x \in B$ but $x \notin A$, then $x \in B \setminus A$. Otherwise $x \in A$. Hence $x \in A \cup (B \setminus A)$.

(b) Proof of $A \cup (B \setminus A) \subset A \cup B$: $\forall x \in A \cup (B \setminus A)$, one has that $x \in A$ or $x \in (B \setminus A) \subset B$, hence $x \in A \cup B$.

Summarising the two directions we conclude that $A \cup B = A \cup (B \setminus A)$ □

Grading. This part is worth 6 points. 2 for the definition of equal set, 2 for LHS to RHS, 2 for RHS to LHS.

(2) Find a counterexample to the statement:

"For all sets A, B, C we have $A \cup B \cup C = (A \setminus B) \cup (B \setminus C) \cup (C \setminus A)$."

Solution. Let $A = B = C$ be a **non-empty** set, then

$$A \cup A \cup A = A \tag{3}$$

$$(A \setminus A) \cup (B \setminus B) \cup (C \setminus C) = \emptyset \cup \emptyset \cup \emptyset = \emptyset \tag{4}$$

But $A \neq \emptyset$ as A is non-empty. This furnishes a counter-example.

Grading. This part is worth 4 points. 2 for giving a reasonable counterexample, 2 for its verification.

Problem 3 [5 points]

This question is related to Problem Set 2 Questions 1.2 and 4.1.

Karampreet has a list of 17 logical statements that each use some number of P, Q and the logic symbols $(\wedge, \vee, \implies, \Leftrightarrow, \neg)$. He (incorrectly) claims that no two of the statements on his list are logically equivalent.

(1) Use the pigeonhole principle to prove that two of the statements on Karampreet's list are logically equivalent.

Solution. Let $\{R_i\}_{1\leq i\leq 17}$ be the 17 statements obtained by compounding P, Q and some connectives. Consider the following table for R_i:

P	Q	R_i
T	T	L_1
T	T	L_2
T	T	L_3
T	T	L_4

The fact that each $L_k \in \{T, F\}$ implies that there are $2^4 = 16$ ways to fill the third column in the above table. Let's consider each possible way as a pigeonhole and each compound statement as a pigeon. Given 16 pigeonholes and 17 pigeons, there will be at least two pigeons (i.e., two compound statements) in the same pigeonhole (i.e., with the same truth values), in other words, these two compound statements are logically equivalent.

Grading. This part is worth 4 points. 1 for the definition of equivalent statements, 2 for correctly choosing the pigeon and the pigeonholes, 1 for application of the pigeonhole principle

(2) Is your proof in the part (1) constructive or non-constructive? Explain your reasoning.

Solution. The proof is a non-constructive proof: although we have shown the existence of two logically equivalent statements based on the pigeonhole principle - which itself is a result of proof by contradiction - we did not give the explicit description of these two statements.

Grading. This part is worth 1 point. All or nothing.