MAT102H5 Y - SUMMER 2020 - QUIZ 3 - SOLUTIONS

Submission

- **You must submit your completed Quiz on Crowdmark by 6:00pm (EDT) Tuesday July 14, 2020**.
- Late submissions will not be accepted.
- You should start uploading your quiz no later than 5:45pm.
- If you require additional space, please insert extra pages.
- You do not need to print out this quiz; you may submit clear pictures/scans of your work on lined paper, or screenshots/PDFs of your work.

Additional Instructions

You must justify and support your solution to each question.

Permitted Resources

During the quiz:

(1) You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
(2) You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, completing problem sets).
(3) Do not use personal notes related to other material (e.g. notes created by studying external websites)
(4) Do not communicate with anyone other than the instructors.
(5) Do not use Piazza.
(6) Do not use any online resources other than Quercus and Crowdmark.

Academic Integrity

By submitting this quiz you affirm that your submission represents entirely your own efforts. You confirm that:

- You have not copied any portion of this work.
- You have not allowed someone else in the course to copy this work.
- You understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

Problem 1 [10 points]

A MAT102 student named Jr. Proofini states:

> "Reflexivity is not needed in the definition of an equivalence relation, because I can always use symmetry and transitivity to prove reflexivity. Here is my proof:
>
> Since $a \sim b$, by symmetry $b \sim a$, now together with transitivity $a \sim a$."

(1) What is the logical mistake in his proof?

Solution. The logical mistake is that Jr. Proofini is assuming that there exists a relation $a \sim b$ in the set. Specifically the line "since $a \sim b$" does not have to be true.

Grading. This is worth 5 points for the main idea.

(2) Give an example of a relation R on the set $A = \{1, 2, 3\}$ that is symmetric and transitive but not reflexive. (You do not need to prove your assertions.)

Solution. The following examples (and any relabellings of them) work:
(a) $R = \emptyset$,
(b) $R = \{(1, 1)\}$
(c) $R = \{(1, 1), (2, 2)\}$
(d) $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Grading. This is 5 points total. 2 points for giving an example of the correct type (i.e. a collection of pairs of A), and 3 points for it being correct.

Problem 2 [10 points]

In class we proved that given an equivalence relation R on a set A, we can obtain a partition of A by equivalence classes determined by R.
Now we consider the other direction: given a partition, we can create an equivalence relation.

Let $k \in \mathbb{N}$. Suppose that $D = \{A_1, A_2, \ldots, A_k\}$ is a given partition of A, i.e.,

$$A = A_1 \cup A_2 \cup \ldots \cup A_k; \text{ and}$$

$$A_i \cap A_j = \emptyset, \quad \forall 1 \leq i \neq j \leq k.$$

We define a new relation R_D (or equivalently $\sim_D$) on A as

$$x \sim_D y \text{ iff } (\exists 1 \leq i \leq k)[x \in A_i \text{ and } y \in A_i].$$

(1) Prove that $\sim_D$ is an equivalence relation on A.

Solution.
Reflexive. Let $a \in A$. By the first property of being a partition, there is an $1 \leq i \leq k$ such that $a \in A_i$. Therefore $a \in A_i$ and $a \in A_i$ and so by definition $a \sim_D a$.

Symmetric. Suppose that $a, b \in A$ are such that $a \sim_D b$. By definition, there is an $1 \leq i \leq k$ such that $a \in A_i$ and $b \in A_i$. Since 'and' is symmetric, we have $b \in A_i$ and $a \in A_i$. So $b \sim_D a$.

Transitive. This is similar to symmetry, except it uses that 'and' is transitive. (Details omitted.)

Grading. This question is worth 6 points. Each part is worth 2.

(2) Let $A = \{5, 6, 7, 8, 9\}$, $D = \{\{5, 6, 7\}, \{8\}, \{9\}\}$, where $A_1 = \{5, 6, 7\}, A_2 = \{8\}$, and $A_3 = \{9\}$. Let $\sim_D$ be the relation constructed in the previous question.

Write down all elements in $\sim_D$ explicitly. (Recall $R_D \subseteq A \times A$.)

Solution. This can be given as a set:

$$R_D = \{(5, 5), (6, 6), (7, 7), (5, 6), (6, 5), (5, 7), (7, 5), (6, 7), (7, 6), (8, 8), (9, 9)\}.$$

Grading. This is worth 4 points. 2 points for a solution of the correct type (a collection of pairs). and 2 points for a correct answer.