Intro to Proofs - Strong Induction

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Learning Objectives

By the end of this session, participants should be able to:

1. State the structure of strong induction.
2. Explain the differences between strong and usual induction.

Motivation

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The final variation of induction we will look at is strong induction. This is a version that is used to prove theorems of the form "Every natural number has a [nice] representation."
It is also used when the induction step depends on many previous steps, and not only the one immediately before it.

1. Proof strategy for strong induction

Strong Induction

If you want to prove a statement of the form "$\forall n \in \mathbb{N}, P(n)$" you can show:

1. $P(1)$ is true,
2. For all $k \in \mathbb{N}, P(1), P(2), \ldots, P(k) \implies P(k+1)$.

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<u>Math answer</u>: The IH is much stronger.

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Question: How is this different that (simple) induction?
Math answer: The IH is much stronger.
CS answer: This requires a lot more memory since you need to remember all your previous work.

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Theorem

Every natural number $n \geq 2$ can be written as a product of primes.

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- $7 =$
- $5! =$

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- $7 = 7$
- $5! = 5 \cdot 3 \cdot 2^3$

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$\boxed{\text{Base, } n = 2}$ Notice 2 is a product of one prime. .

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Induction step. Assume $P(2), P(3), \ldots, P(n)$ are true for a particular $n \in \mathbb{N}$.

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Induction step. Assume $P(2), P(3), \ldots, P(n)$ are true for a particular $n \in \mathbb{N}$.

Case 1: If $n+1$ is prime, then it is the product of one prime.

Case 2: If $n+1$ is not prime, then there are $a, b \in \mathbb{N}$ with $ab = n+1$ and $1 < a, b, < N+1$.

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Case 2: If $n+1$ is not prime, then there are $a, b \in \mathbb{N}$ with $ab = n+1$ and $1 < a, b, < N+1$.
By $P(a)$, a can be written as a product of primes. By $P(b)$, b can be written as a product of primes.

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By $P(a)$, a can be written as a product of primes. By $P(b)$, b can be written as a product of primes.

So $n+1 = ab$ can be written as a product of primes.

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2. Are 3 and 100 prime? 3, yes, so stop. 100, no, $100 = 4 \cdot 25$.

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3. $4 = 2 \cdot 2$ and $25 = 5 \cdot 5$. (and $2, 5$ are prime).

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So

$$300 = [3] \cdot [100] = 3 \cdot [4] \cdot [25] = 3 \cdot 2 \cdot 2 \cdot 5 \cdot 5.$$

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$$300 = [3] \cdot [100] = 3 \cdot [4] \cdot [25] = 3 \cdot 2 \cdot 2 \cdot 5 \cdot 5.$$

Note

To show $P(300)$ we needed to use $P(3)$ and $P(100)$, not $P(299)$.

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I'll complete the proof in a video, or you can read it in the textbook.

Reflection

- What are the differences between simple induction and strong induction?
- Were our proofs about the Fibonacci numbers really "strong" induction? How much memory did they use?
- Why do you think that the binary representation theorem is a good candidate to prove using strong induction?