Intro to Proofs - Binary Representation

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Learning Objectives

By the end of this session, participants should be able to:

1. Produce a valid binary representation of a positive integer, by using recursion.

Strong Induction

If you want to prove a statement of the form "$\forall n \in \mathbb{N}, P(n)$" you can show:

1. $P(1)$ is true,
2. For all $k \in \mathbb{N}, P(1), P(2), \ldots, P(k) \implies P(k+1)$.

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Theorem

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Example

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\begin{aligned}
24 &= 8 + 16 &&= 2^3 + 2^4 \\
48 &= 16 + 32 &&= 2^4 + 2^5 \\
6 &= 2 + 4 &&= 2^1 + 2^2 \\
7 &= 1 + 2 + 4 &&= 2^0 + 2^1 + 2^2
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Major idea: Odd numbers $n+1$ relate to n by adding 2^0. Even numbers $n+1$ relate to $\frac{n+1}{2}$ by removing one from each power.

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Induction Suppose that $1, 2, \ldots, n$ all have valid binary representations for a particular $n \in \mathbb{N}$.

(Continued ...)

3. Proof

Example, $n = 6$

- $6 = 2^1 + 2^2$
- $7 = 2^0 + 2^1 + 2^2$.

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$$n = 2^{a_1} + \ldots + 2^{a_k},$$

where $a_1, a_2, \ldots, a_k \geq 0$.

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where $a_1, a_2, \ldots, a_k \geq 0$. Since n is even, in fact, none of the exponents are 0. Therefore

$$n + 1 = 2^0 + 2^{a_1} + \ldots + 2^{a_k},$$

which is a valid binary representation.

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Example, $n = 47$

$n + 1 = 48 = 2(24)$, and

- $24 = 2^3 + 2^4$
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Case 2, n is odd. So $n + 1$ is even, and there is a (positive) integer m, with $n + 1 = 2m$. By the IH, m has a valid binary representation

$$m = 2^{a_1} + \ldots + 2^{a_k},$$

where $a_1, a_2, \ldots, a_k \geq 0$.
Therefore

$$n + 1 = 2m = 2(2^{a_1} + \ldots + 2^{a_k}) = 2^{a_1+1} + \ldots + 2^{a_k+1},$$

which is a valid binary representation.

Reflection

- Why did we use $a_1, a_2, \ldots, a_k$ for the exponents in this proof? Is there a simpler way to prove this?
- In case 2, how did we know that $1 \leq m \leq n$?
- Are there multiple ways to represent a number in binary?
- Extract a construction from the proof that produces the binary representation of 52.