Introduction to Proofs - Injections, Surjections, and Bijections

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UTM

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Check that a function is 1-1 or onto, and produce counterexamples when it isn't.
2. Give multiple equivalent definitions of injection and surjection.

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Injections will also be used for formally measuring how many elements are in a set (in the countability section).
e.g. Which has more elements: $\mathbb{Z}$ or $(0, +\infty)$?

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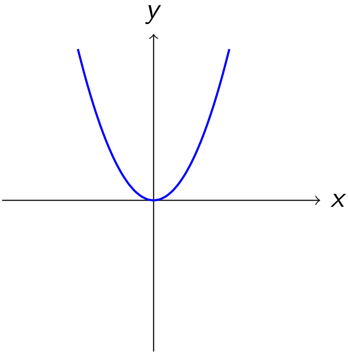
$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is a function.

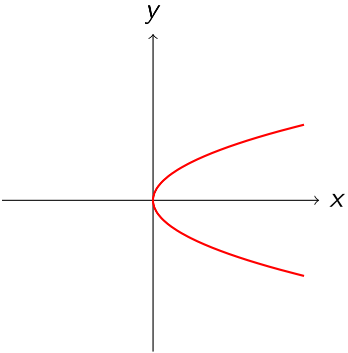
Non example

$g(x) = \pm\sqrt{x}$ is not a function, as $g(2)$ is both $\sqrt{2}$ and $-\sqrt{2}$.

1. Function review

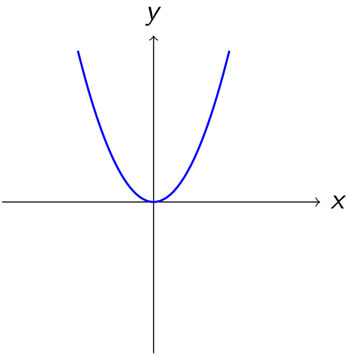
Informally: "A function passes the vertical line test (VLT)."

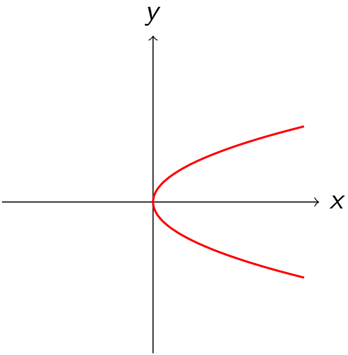




2. Motivation

Question: When can we invert/undo functions?





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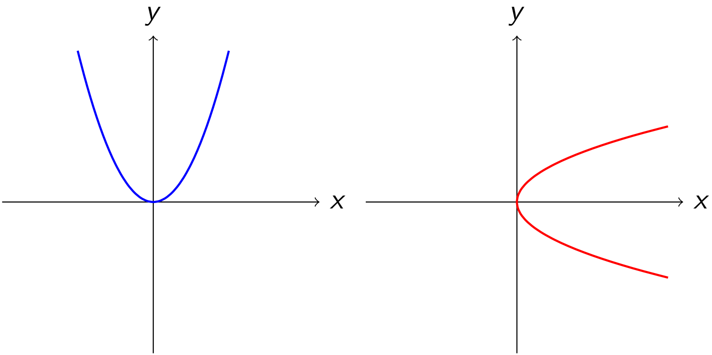

Observation

When the original function fails the "horizontal line test", then the reflection will fail the vertical line test.

3. Definitions

Let $f : A \to B$ be a function.

Definition (Injection)

We say that f is an <u>injection</u> (or is injective, or is one-to-one) if

$$\forall b \in B, \text{ there is at } \underline{\text{most}} \text{ one } a \in A \text{ with } f(a) = b.$$

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A function $f : A \to B$ is not injective if

$$\exists b \in B, \exists a_1, a_2 \in A, \text{ such that } a_1 \neq a_2 \text{ and } f(a_1) = b = f(a_2).$$

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$$\exists b \in B, \forall a \in A, f(a) \neq b.$$

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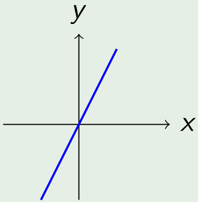
Definition (Bijection)

We say that f is a bijection (or is bijective) if it is one-to-one and onto.

4. Examples

Example 1

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$.



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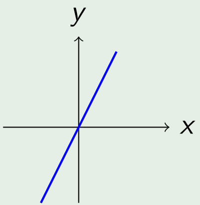

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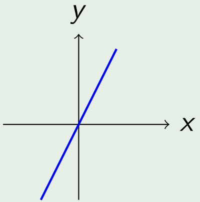

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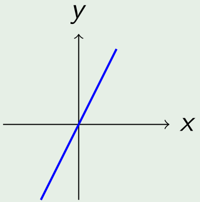

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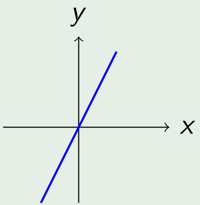

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So f is a bijection.

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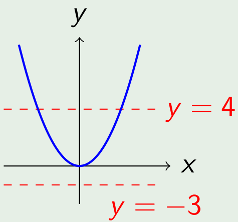
Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$.



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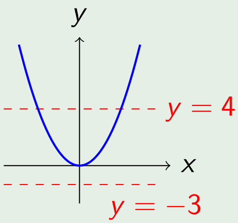

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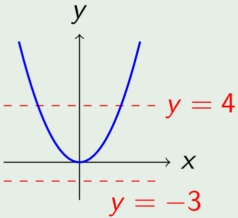

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Not onto As $-3 \in \text{codom}(f) = \mathbb{R}$, but there is not $x \in \mathbb{R}$ with $x^2 = -3$.

5. Exercises

Exercise 1

1. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is onto, but not one-to-one.
2. Find a functions $g : \mathbb{R} \to \mathbb{R}$ that is one-to-one, but not onto.

Exercise 2

1. Show that the polynomial $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = (x - 1)(x - 2)$ is not an injection.
2. Prove that if a polynomial has more than one root, then it is not injective.
3. Prove that even functions are not injective. (Def: $\forall x \in \text{dom}(f) = \mathbb{R}, f(-x) = f(x)$.)
4. Prove that any even degree polynomial will not be an injection.

6. Functions as arrow diagrams

Exercise 3

Write/find/draw four different arrow diagrams $\{1, 2, 3\} \to \{4, 5, 6\}$ as so that

1. One is a bijection.
2. One is not injective.
3. One is not a function.
4. One is not onto, but is one-to-one. [Trick question.]

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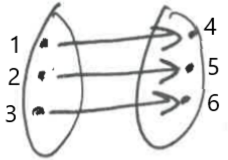

7. Better definitions

Here $f : A \to B$ is a function.

Theorem 1

The following are equivalent (TFAE):

1. f is one-to-one.
2. $\forall a_1, a_2 \in A$: If $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

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   (This is the best version.)

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Theorem 3

The following are equivalent (TFAE):

1. f is a bijection.
2. For each $b \in B$, there is <u>exactly</u> one $a \in A$ with $f(a) = b$.

Reflection

- What is the difference between the horizontal line test and the vertical line test.
- What is the difference between checking that something is a function, versus checking that it is an injection?
- What do the following things look like on an arrow diagram: a surjection, not an injection, not a surjection.