Introduction to Proofs - Inverse functions

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Slides available at: mikepawliuk.ca

Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Produce the inverse of a simple function.

Motivation

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We know that x^2 and $\sqrt{x}$ are "inverse" functions because they "undo" each other.
What does this mean precisely? How do we find inverse functions? When do inverse functions exist?

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Let $f : A \to B$ be a bijection. The <u>inverse of f</u> is the function $g : \quad \to \quad$ that assigns to any $b \in B$ the <u>unique</u> $a \in A$ such that $f(a) = b$.

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2. Examples

Example 1

x^2 and $\sqrt{x}$ are inverses of each other

- $f : [0, \infty) \to [0, \infty)$ with $f(x) = x^2$
- $f^{-1} : [0, \infty) \to [0, \infty)$ with $f(x) = \sqrt{x}$

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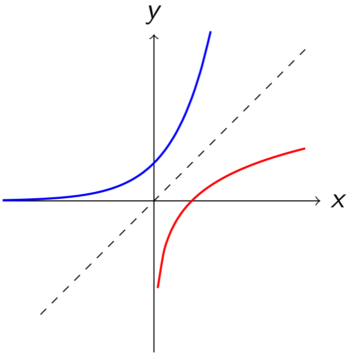
Question: Why did we restrict the domain of x^2 to $[0, \infty)$?
Answer: So that it would pass the horizontal line test (HLT), and its inverse would pass the vertical line test (VLT) and be a function.

2. Examples

Example 2

e^x and $\ln x$ are inverses of each other

- $f : \mathbb{R} \to (0, \infty)$ with $f(x) = e^x$
- $f^{-1} : (0, \infty) \to \mathbb{R}$ with $f^{-1}(x) = \ln x$



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Theorem

Let $f : A \to B$ be a function. The following are equivalent:

1. f is one-to-one and f is onto.
2. f^{-1} exists and is defined on <u>all</u> of B.

Example 2

Let $f : \{1, 2, 3\} \to \{a, b, c\}$ be defined by $f(1) = a$, $f(2) = c$ and $f(3) = b$.

Then $f^{-1} : \quad \to \quad$ is the function

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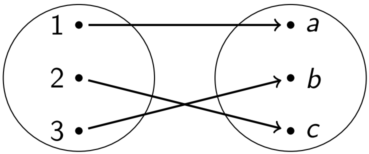

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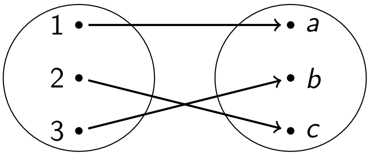

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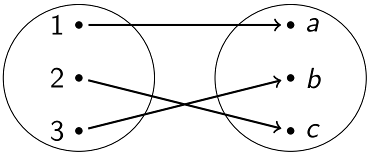

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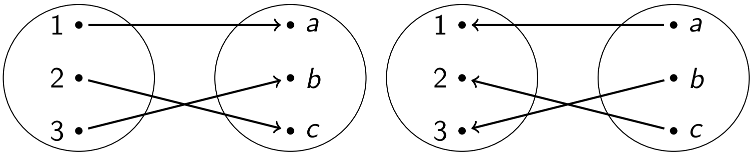

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So $a = b$.

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So $yx + y = x$.

So $y = x - yx$, and $y = x(1-y)$.

So $\dfrac{y}{1-y} = x$. $\boxed{h^{-1}(y) = \dfrac{y}{1-y}}$

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$\boxed{\text{Onto}}$. Let $y \in \mathbb{R} \setminus \{1\}$. So $y \neq 1$.

Let $x = \dfrac{y}{1-y}$. $\boxed{\text{Exercise: Show } x \in \text{dom}(h).}$

Now compute

$$h(x) = h\left(\frac{y}{1-y}\right) = \frac{\frac{y}{1-y}}{\frac{y}{1-y}+1} = \ldots = y$$

Reflection

- What is the role of the HLT and VLT in finding the inverse of a function?
- What is the difference between $f(f^{-1}(y))$ and $f^{-1}(f(x))$?
- How does finding the inverse of a function relate to showing that it is surjective?