Introduction to Proofs - Compositions

Prof Mike Pawliuk

UTM

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Slides available at: mikepawliuk.ca

Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Evaluate if a composition of functions is defined.
2. Prove general facts about compositions of functions, from definitions.

Motivation

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Here we will see how to deal with applying many functions, one after another.

We will also see how to take off your socks.

1. Example

Exercise. Draw the following functions:

1. x^2 [basic function]
2. $3x^2$ [scale up by 3]
3. $-3x^2$ [reflect across x-axis]
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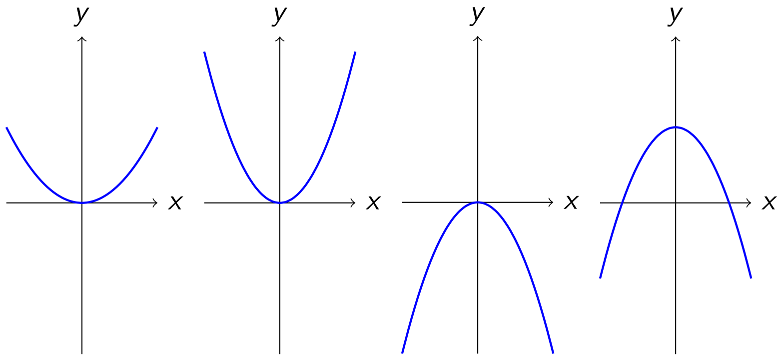

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This function was composed of four simple functions:

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\begin{aligned}
a(x) &= x^2 \\
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Exercise: Show that $a(b(x)) \neq b(a(x))$.

2. Definition

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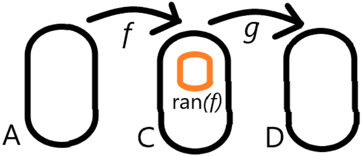

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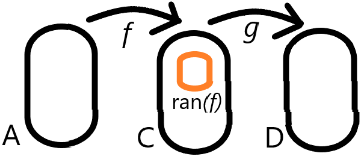

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$$\text{ran}(f) \subseteq \text{dom}(g).$$

Usually we'll actually have $\text{codom}(f) \subseteq \text{dom}(g)$.

3. The order of compositions really matters

Example

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be defined by $f(n, m) = n - m$.
Let $g : \mathbb{Z} \to \mathbb{R}$ be defined by $g(x) = \sqrt{|x|}$.

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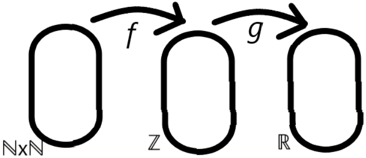

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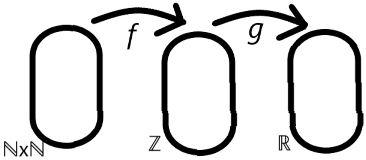
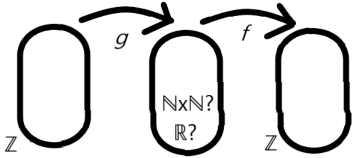


$$f \circ g = ?$$



4. Functions and compositions

This is a fundamental result in math. Look for it in every math course.

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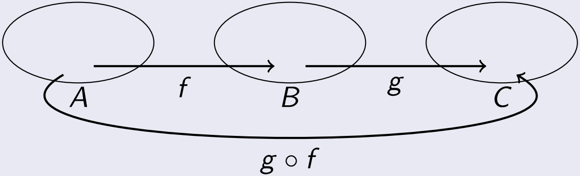
Proof of 1 is an important exercise.

4. The composition of any two surjections is a surjection.

Proof.

Let $f : A \to B$ and $g : B \to C$ be surjections.



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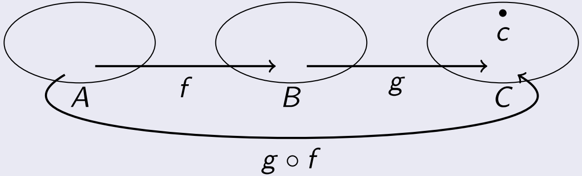

Let $c \in C$. Goal: Find an $a \in A$ with $g \circ f(a) = c$.

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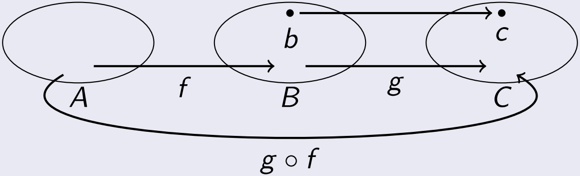

Let $c \in C$. | Goal: Find an $a \in A$ with $g \circ f(a) = c$. |

Because g is a surjection, there is a $b \in B$ with $g(b) = C$.

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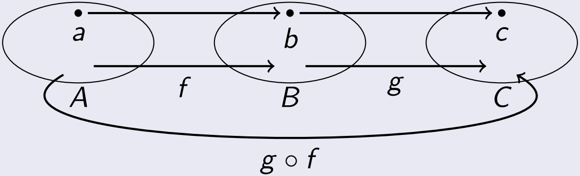

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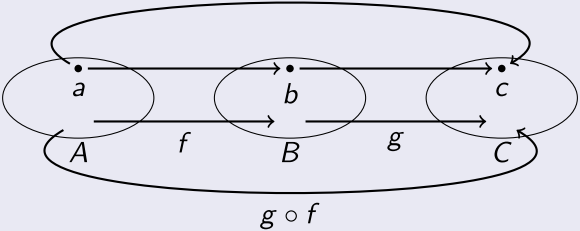

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Because g is a surjection, there is a $b \in B$ with $g(b) = C$.

Because f is a surjection, there is an $a \in A$ with $f(a) = b$.

Note $g \circ f(a) = g(f(a)) = g(b) = c$.

□

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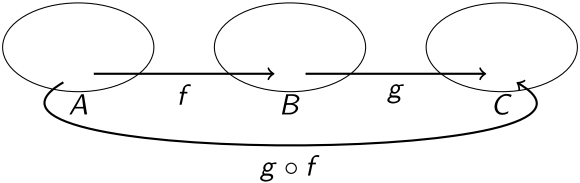

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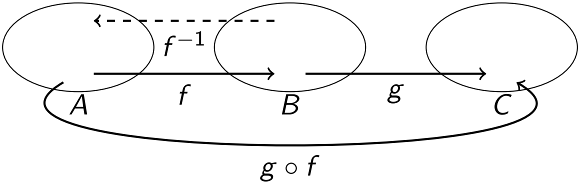

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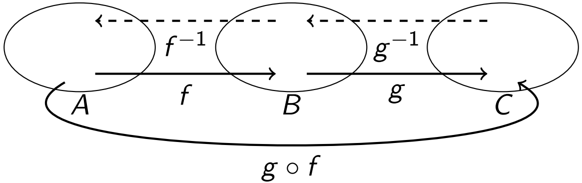

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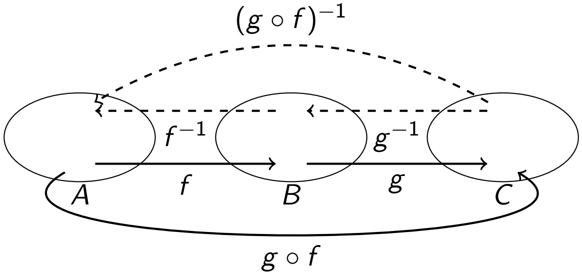

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Note 2. $f^{-1} \circ g^{-1}(c) = f^{-1}(b) = a$.

So $(g \circ f)^{-1}(c) = f^{-1} \circ g^{-1}(c)$.

Reflection

- Does the order in which you compose functions matter?
- For $g \circ f$, why do we need $\text{ran}(f) \subseteq \text{dom}(g)$?
- How do socks and shoes relate to inverse functions?