Introduction to Proofs - Cardinality - Definitions

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Define relative cardinality
2. State and prove basic abstract results about cardinality.

1. Motivation

Example 1

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Example 2

Are there more people or chairs in this picture?

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Conclusion

We have an assignment of each person to a different chair, and there are chairs left over. So there are more chairs than people.

1. Motivation

Observations

1. We don't need numbers to do this.
2. We can measure relative quantities using the language of functions (injections, surjections, bijections).

Example 2

What type of function is the assignment of people to chairs? (Domain, codomain, injective, surjective, bijective?)

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Conclusion

It is a function $f : \{\text{people in picture}\} \to \{\text{chairs in picture}\}$ that is an injection (but not a surjection).

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Let A and B be two sets. We define

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Informally:

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Informally: $|A|$ is the "number of elements in A". It has nothing to do with absolute values.

We will not use this as our definition.

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<u>Note.</u> This property is like transitivity for relative cardinalities.

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1. For all sets A, $|A| \leq |A|$.
2. For all sets A, $|A| = |A|$.
3. For all sets A, B, if $|A| = |B|$, then $|B| = |A|$.
4. For all sets A, B, C, if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
5. For all sets A, B, if $|A| = |B|$, then $|A| \leq |B|$.

Exercise: Prove all of these using the definitions.

4. Cantor-Schroeder-Bernstein

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1. Construct an injection $f : (-2, 2) \to [-2, 2]$.
2. Construct an injection $g : [-2, 2] \to (-2, 2)$.
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Question: Is the converse true?

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Question: Is the converse true?
No! For example, $|\{1,2\}| \leq |\{4,5,6\}|$, but $\{1,2\} \not\subseteq \{4,5,6\}$.

Finite CSB

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Exercise. This proof is much easier than the general CSB. (Where did you use finiteness?)

Reflection

- What is the definition of $|A| \leq |B|$?
- What are the sets A where $|A| = |\emptyset|$?
- What is the role of surjections in determining relative cardinality?