Introduction to Proofs - Cardinality - Examples

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Determine the relative cardinality of two sets.
2. Prove that all intervals have the same cardinality.

1. Motivation

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Now that we know the definitions of relative cardinality, how do we actually compare cardinalities of actual sets?

2. Examples

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Let $f : \mathbb{N} \to E$ be defined by $f(n) = 2n$.
This is clearly a bijection. (Exercise?) □

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Let $A = \{1, 2, 3, \ldots, 2020\}$. Show that $|A| \leq |\mathbb{R}|$ and that $|A| \neq |\mathbb{R}|$.

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Let $y = \max(\{f(1), f(2), \ldots, f(2020)\}) + 1$. Note that $y > f(x)$ for all $1 \leq x \leq 2020$. So $y \notin \text{range}(f)$. □

3. Intervals

Goal

Let $a < b$ and $c < d$ be real numbers. Then $|[a, b]| = |[c, d]|$.

Exercises.

1. Show $|[0, 1]| = |[1, 2]|$.
2. Show $|[0, 1]| = |[0, 2]|$.
3. Show $|[0, 1]| = |[1, 3]|$.
4. Show $|[0, 1]| = |[a, b]|$.
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4. Show $|[0, 1]| = |[a, b]|$. $f(x) = (b - a)x + a$
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4. Show $|[0, 1]| = |[a, b]|$. $f(x) = (b - a)x + a$
5. Conclude $|[a, b]| = |[c, d]|$. Since $|[c, d]| = |[0, 1]|$.

Corollary: $|(a, b)| = |(c, d)|$.

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Exercise. Show that the following have the same cardinalities. Assume $a < b$.

1. $\mathbb{R}$
2. (a, b)
3. $[a, b]$
4. (a, ∞)
5. $[a, \infty)$
6. $(-\infty, b)$
7. $(-\infty, b]$

Reflection

- If two intervals have the same cardinality, do they have to have the same length?
- Wait. We showed that the evens and the natural numbers have the same cardinality. Shouldn't it be $|E| < |\mathbb{N}|$ since the evens are a subset of $\mathbb{N}$?