Introduction to Proofs - Countability - Intro

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. State the definition of countable.
2. Use Cantor-Schroeder-Bernstein to show that some sets are countable.

1. Motivation

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One special case of infinite cardinality is when a set has the same cardinality as $\mathbb{N}$. We think of $\mathbb{N}$ as being a "small" infinite set. Our goal is to find other sets that have the same cardinality as $\mathbb{N}$.

2. Hilbert's Hotel

Hilbert's Hotel 1

The mathematician Hilbert has a fantastic hotel with one room for every natural number.

A farmer arrives at the hotel, and are sad to see that every room in the hotel is full! However, Hilbert, by rearranging his guests, finds a way so that every old guest has a room, and the farmer also has a room.

How does he do this?

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How does he do this?

One answer: Hilbert asks every current guest to move into the room one up (i.e. guests in room n go to room $n+1$). Then he gives room 1 to the farmer.

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One answer: Hilbert asks every current guest to move into the room 2020 up (i.e. guests in room n go to room $n + 2020$). Then he gives rooms 1 through 2020 to the farmers.

2. Hilbert's Hotel

Hilbert's Hotel 3

The hotel is so lovely, that every current guest phones one friend and tells them they have to get a room at Hilbert's hotel.
Hoe does Hilbert find a way to accommodate all these new friends?

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One answer: Hilbert asks every current guest to move into the room that is double their current room (i.e. guests in room n go to room $2n$). Then each guest's friend can go in the room one above them $(2n + 1)$.

Hilbert's Hotel 4

One day, a (large) soccer team shows up to the hotel. Each player wears a jersey with a different real number on it. There is a π player and a $\sqrt{2}$ player and a $\frac{7}{2}$ player; all real numbers are represented.

Can Hilbert find a way to accommodate the whole team into his hotel?

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Answer: We'll see this later!

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Lemma

The following are equivalent:

1. A is countable.
2. There is a bijection $g : \mathbb{N} \to A$
3. A can be enumerated using $\mathbb{N}$ in a list as $a_1, a_2, a_3, \ldots$

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Note: If you have a bijection $g : \mathbb{N} \to A$, then one enumeration is $g(1), g(2), g(3), \ldots$

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Example 2

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Proof.

Let $g : \mathbb{N} \to \{2, 4, 6, 8, \dots\}$ be defined by $g(n) = 2n$. This is clearly a bijection. $\square$

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We can write this as a list:

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

And after thinking for a bit, this is given by the bijection $g : \mathbb{N} \to \mathbb{Z}$

$$g(x) =$$

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$$g(x) = \begin{cases} -(n-1)/2 & n \text{ is odd} \\ n/2 & n \text{ is even} \end{cases}$$

□

4. Other Examples

Other examples

The following sets are countable.

1. $\mathbb{N}$
2. $\mathbb{Z}$
3. $\mathbb{Q}$
4. $\mathbb{N} \times \mathbb{N}$, $\mathbb{Z} \times \mathbb{Z}$.
5. Every infinite subset of a countable set. ("Countable is the smallest size of infinity.")

We'll prove these later.

5. Other results

Theorem

The following sets are countable.

1. The collection of all finite strings of 0s and 1s is countable.
2. If A, B are countable, $A \cup B$ is countable
3. If A, B are countable, then $A \times B$ is countable.
4. If A_n is countable for each $n \in \mathbb{N}$, then $A_1 \cup A_2 \cup A_3 \cup \ldots$ is countable.

Challenge exercise. Prove these results.

Reflection

- Which of these gives an enumeration/listing of A: a bijection $f : A \to \mathbb{N}$ or $g : \mathbb{N} \to A$.
- Are there countably many prime numbers?
- Are there countably many powers of 10?