# Introduction to Proofs - Countability - Diagonalization

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# Learning Objectives

By the end of this video, participants should be able to:

1. Define uncountability of a set.
2. Apply Cantor's Diagonalization to a list of real numbers (finite or countable).

# 1. Motivation

### Motivation

We have seen that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}$ are all countable sets. Is every set either finite or countable?

No. We will see that $\mathbb{R}$ is infinite and not countable. We will use a non-trivial technique called "diagonalization".

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<u>Idea</u>. We think of:

- finite sets as extremely small,
- countable sets as small and infinite,
- uncountable sets as large and infinite.

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5. Choose $f(a_{n+1})$ to be any element of $B \setminus \{f(a_1), f(a_2), \ldots, f(a_n)\}$.

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Note that if $i \neq j$, then $f(a_i) \neq f(a_j)$ by construction. So $f$ is an injection. □

# 3. Diagonalization

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We will use a method called Cantor's diagonalization.

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$$x = 0.\boxed{1}00$$
$$y = 0.5\boxed{0}0$$
$$z = 0.90\boxed{0}$$
$$a = 0.217$$

- $a \neq x$ because the first digit of $a$ is 2, and the first digit of $x$ is 1.
- $a \neq y$ because the second digit of $a$ is 1, and the first digit of $y$ is 0.
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**Reflection**

- How many digits do we need to diagonalize a list with 2020 numbers?
- How many digits do we need to diagonalize a list with countably many numbers?
- Will this process always give us a real number between 0 and 1?
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# 4. $[0, 1]$ is uncountable

**Theorem (Cantor)**

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## Proof.

Clearly $[0, 1]$ is infinite. Suppose that $f : \mathbb{N} \to [0, 1]$ is a function. We will show that it is not a surjection by constructing an $a \in [0, 1]$ that is not equal to any $f(n)$.

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f(1) &= 0.\boxed{x_{11}}x_{12}x_{13}\ldots \\
f(2) &= 0.x_{21}\boxed{x_{22}}x_{23}\ldots \\
f(3) &= 0.x_{31}x_{32}\boxed{x_{33}}\ldots \\
&\vdots \\
a &= 0.p(x_{11})p(x_{22})p(x_{33})
\end{aligned}
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Here $p : \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \to \{0, 1\}$ is defined by $p(0) = 1$, and $p(1) = p(2) = \ldots = p(9) = 0$.
So $a \in [0, 1]$ and $a \neq f(n)$ for each $n \in \mathbb{N}$.

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# Reflection

- What does diagonalizing a list produce?
- What is the role of the $p$ function in Cantor's diagonalization proof?
- How did we know that $a \neq f(1)$ in Cantor's diagonalization proof?
- Wait, there are different sizes of infinity?!