Introduction to Proofs - Countability - Diagonalization

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Learning Objectives

By the end of this video, participants should be able to:

1. Define uncountability of a set.
2. Apply Cantor's Diagonalization to a list of real numbers (finite or countable).

1. Motivation

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We have seen that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}$ are all countable sets. Is every set either finite or countable?

No. We will see that $\mathbb{R}$ is infinite and not countable. We will use a non-trivial technique called "diagonalization".

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Idea. We think of:

- finite sets as extremely small,
- countable sets as small and infinite,
- uncountable sets as large and infinite.

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Since A is countable, we can enumerate it as $A = \{a_i : i \in \mathbb{N}\}$. Define an injection $f : A \to B$ recursively by:

1. $f(a_1)$ is any element of B.

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4. ...
5. Choose $f(a_{n+1})$ to be any element of $B \setminus \{f(a_1), f(a_2), \ldots, f(a_n)\}$.

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4. $\ldots$
5. Choose $f(a_{n+1})$ to be any element of $B \setminus \{f(a_1), f(a_2), \ldots, f(a_n)\}$.

Note that if $i \neq j$, then $f(a_i) \neq f(a_j)$ by construction. So f is an injection. □

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We will use a method called Cantor's diagonalization.

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$$x = 0.\boxed{1}00$$
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$$a = 0.21$$

- $a \neq x$ because the first digit of a is 2, and the first digit of x is 1.
- $a \neq y$ because the second digit of a is 1, and the first digit of y is 0.

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$$x = 0.\boxed{1}00$$
$$y = 0.5\boxed{0}0$$
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$$a = 0.217$$

- $a \neq x$ because the first digit of a is 2, and the first digit of x is 1.
- $a \neq y$ because the second digit of a is 1, and the first digit of y is 0.
- $a \neq z$ because the third digit of a is 7, and the third digit of z is 0.

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$$a = 0.101$$

Reflection

- How many digits do we need to diagonalize a list with 2020 numbers?
- How many digits do we need to diagonalize a list with countably many numbers?
- Will this process always give us a real number between 0 and 1?
- How can we formalize, or automate, the idea of choosing a different digit?

4. $[0, 1]$ is uncountable

Theorem (Cantor)

$[0, 1]$ is uncountable.

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Proof.

Clearly $[0, 1]$ is infinite. Suppose that $f : \mathbb{N} \to [0, 1]$ is a function. We will show that it is not a surjection by constructing an $a \in [0, 1]$ that is not equal to any $f(n)$.

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$$\begin{aligned} f(1) &= 0.\boxed{x_{11}}x_{12}x_{13}\dots \\ f(2) &= 0.x_{21}\boxed{x_{22}}x_{23}\dots \\ f(3) &= 0.x_{31}x_{32}\boxed{x_{33}}\dots \\ &\vdots \end{aligned}$$

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\end{aligned}
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Here $p:\{0,1,2,3,4,5,6,7,8,9\}\to\{0,1\}$ is defined by $p(0)=1$, and $p(1)=p(2)=\ldots=p(9)=0$.

$\square$

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Here $p : \{0,1,2,3,4,5,6,7,8,9\} \to \{0,1\}$ is defined by $p(0) = 1$, and $p(1) = p(2) = \ldots = p(9) = 0$.
So $a \in [0,1]$ and $a \neq f(n)$ for each $n \in \mathbb{N}$.

□

Reflection

- What does diagonalizing a list produce?
- What is the role of the p function in Cantor's diagonalization proof?
- How did we know that $a \neq f(1)$ in Cantor's diagonalization proof?
- Wait, there are different sizes of infinity?!